Math 121, Final Test, 14 December 2006

Name. Hints and Answers

Instructions. Do 20 of the following 22 questions. Each question is worth 5pts. Please show all appropriate work in order to obtain maximal credit. Good Luck.

1. Write the complex number \( \frac{1 + 3i}{3 + 2i} \) in standard form.

\[
\frac{1+3i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{3-2i+9i-6i^2}{9+4} = \frac{9+7i}{13} = \frac{9}{13} + \frac{7}{13}i
\]

2. The maximum exercise heart rate in beats per minute is given by

\[
\text{Maximum exercise heart rate} = 0.85(220 - a)
\]

where \(a\) is age in years.

(a) Find the maximum exercise heart rate of a 25 year old.

\[
\text{Max E.H.R.} = 0.85(220-25) \approx 165.75 \approx 166 \text{ BPM}
\]

(b) Find the age of a person whose maximum exercise heart rate is 150 beats per minute.

\[
150 = 0.85(220-a) \Rightarrow 220-a = \frac{150}{0.85} \Rightarrow a = 220 - \frac{150}{0.85} \approx 43.529 \text{ years}
\]

3. Solve the inequality \( x^2 > 10 - 3x \). Write your answer in interval notation.

\[
\Rightarrow x^2 + 3x - 10 > 0
\]

\[
\Rightarrow (x+5)(x-2) > 0
\]

Critical points: \( x=-5 \) and \( x=2 \)

\[
\begin{array}{c|cccccc}
& x+5 & - & - & 0 & + & + & + & + & + \\
\hline
x-2 & - & - & - & - & - & - & 0 & + & + \\
\end{array}
\]

\[
\begin{array}{c|cccccc}
(x+5)(x-2) & + & + & 0 & - & - & - & 0 & + & + \\
\end{array}
\]

\(
\therefore x < -5 \text{ or } x > 2
\)

\[
(-\infty, -5) \cup (2, \infty)
\]
4. Solve the radical equation \( \sqrt{2x - 5} - \sqrt{x + 1} = -1 \). Check all proposed solutions.

\[
\begin{align*}
\sqrt{2x - 5} &= \sqrt{x + 1} - 1 \\
\Rightarrow 2x - 5 &= (x + 1) - 2\sqrt{x + 1} + 1 \\
\Rightarrow x - 7 &= -2\sqrt{x + 1} \\
(\sqrt{x - 7})^2 &= 4(x + 1) \\
\Rightarrow x^2 - 14x + 49 &= 4x + 4 \\
\Rightarrow x^2 - 18x + 45 &= 0
\end{align*}
\]

\( \Rightarrow (x - 3)(x - 15) = 0 \)

- Proposed solutions
  \( x = 3 \quad \text{or} \quad x = 15 \)

Check: \( x = 3 \):
\( \sqrt{1} - \sqrt{4} = -1 \)
\( x = 15 \):
\( \sqrt{25} - \sqrt{16} = -1 \)

\[ x = 3 \]

5. (a) Find the midpoint of \((1, -3)\) and \((5, -7)\).

\[ M = \left( \frac{1+5}{2}, \frac{-3-7}{2} \right) = (3, -5) \]

(b) Find the distance between \((1, 0)\) and \((4, -5)\). (Leave answer in radical form)

\[ D = \sqrt{(4-1)^2 + (-5-0)^2} = \sqrt{9 + 25} = \sqrt{34} \]

6. Find the equation of the line through the points \((-2, 4)\) and \((0, -1)\). Write your answer in slope-intercept form.

\[
m = \frac{y - (-1)}{x - 0} = -\frac{5}{2}
\]

\[ y + 1 = -\frac{5}{2}(x - 0) \]

\[ y = -\frac{5}{2}x - 1 \]

7. Let \( P(x) = 5x^6 - 5x^5 + 4x^2 + 7x + 6 \); use the Rational Zero Theorem to list all possible rational zeros of \( P(x) \).

Possible zeros: \( \pm 1, \pm 2, \pm 3, \pm 6 \)
\[ \pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{3}{5}, \pm \frac{6}{5} \]
8. Consider the quadratic function \( f(x) = -3x^2 + 12x + 5 \).

(a) Does the graph of \( y = f(x) \) open upward or downward?
\[ a_2 = -3 < 0 \quad \therefore \text{ opens downward} \]

(b) Does \( f \) have a maximum or does \( f \) have a minimum?
\[ \text{Maximum} \]

(c) Find the vertex of the graph \( y = -3x^2 + 12x + 5 \).
\[ h = \frac{-b}{2a} = -\frac{12}{2(-3)} = \frac{12}{6} = 2 \]
\[ k = f(2) = -3(2^2) + 12(2) + 5 = 12 + 5 = 17 \]
\[ \text{Vertex is: } (2, 17) \]

(d) Find the range of the function \( f(x) \). Write your answer in interval notation.
\[ (-\infty, 17] \]

9. A rectangle has a length of \( l \) feet and a perimeter of 80 feet.

(a) Write the width \( w \) of the rectangle as a function of its length.
\[ 2l + 2w = 80 \quad \Rightarrow \quad 2w = 80 - 2l \quad \Rightarrow \quad w = 40 - l \]

(b) Write the area \( A \) of the rectangle as a function of its length.
\[ A = w \cdot l = (40 - l)l = -l^2 + 40l \]
\[ A = -l^2 + 40l \]

10. Write \( P(x) = x^4 + x^3 - 2x^2 + 4x - 24 \) as a product of its linear factors given that 2 and -3 are zeros of \( P(x) \).
\[
\begin{array}{c|cccc}
  & 1 & 1 & -2 & 4 \\
-3 & & & & \\
\hline
  & 2 & 6 & 8 & 24 \\
\hline
  & 1 & 3 & 4 & 12 & 10 \\
\hline
  & -2 & 0 & -12 \\
\hline
  & 1 & 0 & 4 & 10 \\
\end{array}
\]
\[ \therefore \quad x^2 + 4 = 0 \quad \Rightarrow \quad x = \pm 2i \]
11. Consider the polynomial \( P(x) = (x - 2)^3(x + 1)^5(x - 3)^2 \). Find its \( x \)- and \( y \)-intercepts; describe its far right and far left behavior; for each intercept determine whether its graph crosses or merely touches the \( x \)-axis. Then make a rough sketch of the graph of \( P(x) \).

\[
\begin{align*}
x\text{-ints:} & \quad (2,0) \quad \text{crosses} \\
& \quad (-1,0) \quad \text{crosses} \\
& \quad (3,0) \quad \text{touches} \\
y\text{-int:} & \quad (-2)^3(1)(-3)^2 = -72 \\
& \quad (0, -72) \\
deg P & = 10 \\
\text{up for right, up for left.}
\end{align*}
\]

12. The rational function \( R(x) = \frac{x - 1}{x + 2} \) has horizontal asymptote \( y = 1 \) and vertical asymptote \( x = -2 \). Graph \( R(x) \) along with its asymptotes. Make sure to: plot and label the \( x \)-intercept and \( y \)-intercept; show the behavior of \( R(x) \) on each side of the vertical asymptote; plot additional points as necessary; and indicate the long term behavior of \( R(x) \).

13. Consider the following systems of equations. For each system, find the values of \( k \) for which there are (if possible), (i) one solution, (ii) no solution, (iii) infinitely many solutions?

(a) \[
\begin{align*}
3x + 4y - z & = 10 \\
5y + 2z & = 0 \\
kz & = 0
\end{align*}
\]

(b) \[
\begin{align*}
3x - 6y - z & = 5 \\
y + z & = 19 \\
kz & = 100
\end{align*}
\]

(i) \( k \neq 0 \)
(ii) no value of \( k \)
(iii) \( k = 0 \)

(i) \( k \neq 0 \)
(ii) \( k = 0 \)
(iii) not possible.
14. Find and then solve the system of equations that is needed to find \( a, b \) and \( c \) in the equation of the parabola \( y = ax^2 + bx + c \) whose graph passes through the points \((2, 4), (0, -4)\) and \((-2, 4)\).

\[
\begin{align*}
\text{Equations:} & \quad 4a + 2b + c = 4 \\
& \quad c = -4 \\
& \quad 4a - 2b + c = 4
\end{align*}
\]

\[
\begin{align*}
\therefore \quad c = -4 \Rightarrow & \quad 4a + 2b = 8 \\
& \quad 8a = 16 \\
\therefore \quad a = 2, \quad b = 0
\end{align*}
\]

An so \( a = 2, \quad b = 0, \quad c = -4 \), or:

\[
y = 2x^2 - 4
\]

15. Find the inverse of \( A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & -1 \\ -4 & -13 & 4 \end{bmatrix} \) if it exists.

\[
\begin{align*}
\begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & -1 \\ -4 & -13 & 4 \end{bmatrix} & \xrightarrow{\frac{1}{2}R_1 + R_3} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\
& \xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\
& \xrightarrow{R_1 - 3R_2 + R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\
\therefore \quad A^{-1} & = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{(Check by multiplying } A \cdot A^{-1} \text{ to get } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix})
\end{align*}
\]

16. Find and simplify the term of \((2x - 3y)^{11}\) that contains \(x^6\) (do not find the full expansion).

\[
\begin{align*}
\frac{11!}{6! \cdot 5!} \cdot (2x)^6 \cdot (-3y)^5 & = \frac{(11 \cdot 10 \cdot 9 \cdot 8 \cdot 7)}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot (64x^6)(-243y^5) \\
& = -\frac{11 \cdot 6 \cdot 7 \cdot 64 \cdot 243}{120} x^6 y^5 \\
& = -7,185,024 x^6 y^5
\end{align*}
\]
17. Find the inverse function of \( f(x) = (x - 2)^\frac{5}{3} + 3 \). Verify that the function you have found is the inverse function to \( f \).

\[
\begin{align*}
y &= (x - 2)^\frac{5}{3} + 3 \\
x &= (y - 2)^\frac{5}{3} \\
\Rightarrow & \quad x - 3 = (y - 2)^\frac{5}{3} \\
\Rightarrow & \quad (x - 3)^{\frac{3}{5}} = y - 2 \\
\Rightarrow & \quad y = 2 + (x - 3)^{\frac{3}{5}}
\end{align*}
\]

\[
\begin{align*}
f^{-1}(x) &= (x - 3)^{\frac{3}{5}} + 2 \\
\text{Check:} \\
\quad f(f^{-1}(x)) &= (f^{-1}(x) - 2)^{\frac{5}{3}} + 3 \\
&= ((x - 3)^{\frac{3}{5}} + 2 - 2)^{\frac{5}{3}} + 3 \\
&= x - 3 + 2 = x
\end{align*}
\]

18. Write the equation \( y = \log_2(x + 3) \) in exponential form, and then sketch its graph.

\[
x + 3 = 2^y \\
\Rightarrow \quad x = 2^y - 3
\]

19. Write \( \log_b \left( \frac{\sqrt[3]{x^2 z^3}}{y^5} \right) \) in terms of \( \log_b x \), \( \log_b y \) and \( \log_b z \).

\[
\begin{align*}
\log_b \left( \frac{\sqrt[3]{x^2 z^3}}{y^5} \right) &= \frac{1}{2} \log_b x^2 + \frac{1}{3} \log_b z^3 - 5 \log_b y \\
&= \log_b x + \frac{2}{2} \log_b z - 5 \log_b y
\end{align*}
\]
20. The population of a town is given by \( P(t) = 37,000(1.08)^t \) where \( t \) is given in years, and \( t = 0 \) now.

(a) What is the present population of the town?

\[ 37,000 \]

(b) What will the population be in 4 years?

\[ P(4) = 37,000(1.08)^4 \approx 50,338. \]

(c) The town will need to replace its sewage treatment plan when the population is 70,000. In how many years will that be?

\[
70,000 = 37,000(1.08)^t \Rightarrow \ln \left( \frac{70}{37} \right) = t \ln(1.08) \Rightarrow t = \frac{\ln(70) - \ln(37)}{\ln(1.08)} \approx 8.284 \text{ years}.
\]

21. Solve the equation \( 1 + \log(3x - 1) = \log(2x + 1) \).

\[
\Rightarrow \quad 1 = \log \left( \frac{2x+1}{3x-1} \right) \Rightarrow \quad 10 = \frac{2x+1}{3x-1} \Rightarrow \quad 30x - 10 = 2x + 1
\]

\[ \Rightarrow \quad 28x = 11 \Rightarrow \quad x = \frac{11}{28} \]

22. Suppose $20,000 is invested at an annual interest rate of 7% compounded monthly. Use the formula \( A = P \left(1 + \frac{r}{n}\right)^{nt} \) to help you solve the following.

(a) How much will it be worth after 10 years?

\[ A = 20,000 \left(1 + \frac{0.07}{12}\right)^{12 \cdot 10} \approx 40,193.23 \]

(b) How long will it take until the investment is worth $100,000?

\[ 100,000 = 20,000 \left(1 + \frac{0.07}{12}\right)^{12t} \Rightarrow \quad t = \frac{\ln 5}{12 \ln \left(1 + \frac{0.07}{12}\right)} = 23.059 \text{ years} \]

\[ \Rightarrow \quad \ln 5 = 12t \ln \left(1 + \frac{0.07}{12}\right) \]