1. The augmented matrix \[
\begin{bmatrix}
1 & 2 & 0 & -3 \\
0 & 1 & -3 & 0 \\
0 & 0 & k & -4
\end{bmatrix}
\] represents a system of equations in the variables \(x, y, z\).

(a) For what values of \(k\) is there no solution?

\textbf{Answer.} \(k = 0\)

(b) For what values of \(k\) is there exactly one solution?

\textbf{Answer.} \(k \neq 0\)

(c) Find the solution to this system of equations if \(k = -2\).

\textbf{Answer.} \(-2z = -4\) implies \(z = 2\). Then \(y - 3z = 0\) implies \(y = 3z = 6\) and \(x + 2y = -3\) implies \(x = -3 - 2y = -3 - 12 = -15\). Thus the solution is\(x = -15, \ y = 6, \ z = 2\).

2. Let \(A = \begin{bmatrix} 2 & -1 & 4 \\ -3 & 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 3 & -1 & 1 \\ -2 & 2 & 0 \end{bmatrix}\). Find the following, if possible, if an operation is not possible, state why it is not possible.

(a) \(BC\) (b) \(CB\) (c) \(2A - 2C\)

\textbf{Answer.} (a) \(BC = \begin{bmatrix} -3 & 5 & 1 \\ 8 & -4 & 2 \end{bmatrix}\).

(b) \(CB\) does not exist because the number of columns of \(C\) is not equal to the number of rows of \(B\).

(c) \(2A - 2C = \begin{bmatrix} -2 & 0 & 6 \\ -2 & -2 & 0 \end{bmatrix}\).

3. Consider the following system of equations:
\[
\begin{align*}
x + y + 2z &= 1 \\
2x + 3y + 3z &= -2 \\
3x + 3y + 7z &= 3
\end{align*}
\]

Solve this system using the fact that the inverse of \[
\begin{bmatrix}
1 & 1 & 2 \\
2 & 3 & 3 \\
3 & 3 & 7
\end{bmatrix}
\] is \[
\begin{bmatrix}
12 & -1 & -3 \\
-5 & 1 & 1 \\
-3 & 0 & 1
\end{bmatrix}
\].

\textbf{Answer.} Multiply the inverse of \(A\) by the matrix of constants as follows:
\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix} 12 & -1 & -3 \\
-5 & 1 & 1 \\
-3 & 0 & 1
\end{bmatrix} \begin{bmatrix} 1 \\
-2 \\
3
\end{bmatrix} = \begin{bmatrix} 5 \\
-4 \\
0
\end{bmatrix}
\]

The solution is \((x, y, z) = (5, -4, 0)\).
4. Consider the matrices 
\[ A = \begin{bmatrix} 12 & -1 & -3 \\ -5 & 1 & 1 \\ -3 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -0 & -3 & 2 \\ -1 & 3 & 3 & 0 \\ -2 & 4 & -1 & 2 \end{bmatrix} \]

(a) Let \( C = AB \). Determine the dimensions of \( C \), and find \( c_{23} \) (you don’t need to compute the entire product).

**Answer.** \( C \) has 3 rows and 4 columns.

\[ c_{23} = \begin{bmatrix} -5 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix} = -5(-3) + 1(3) + 1(-1) = 17. \]

(b) Does \( D = BA \) exist? If so, determine the dimensions of \( D \) and find \( d_{31} \).

**Answer.** This product does not exist since the number of columns of \( B \) is not equal to the number of rows of \( A \).

5. (a) Let \( A \) be a 2 by 3 matrix, how many rows and how many columns must \( B \) have so that \( C = AB \) is a 2 by 8 matrix?

(b) If \( A \) is an \( m \) by \( n \) matrix and \( B \) is an \( n \) by \( p \) matrix, what are the dimensions of the product \( AB \)?

(c) If \( A \) is an \( m \) by \( n \) matrix and \( B \) is a \( r \) by \( s \) matrix. Under what condition(s) will \( AB \) exist? Under what condition(s) will \( BA \) exist?

**Answer.** (a) \( B \) must be a 3 by 8 matrix.

(b) \( AB \) will be a \( m \) by \( p \) matrix.

(c) \( AB \) will exist if \( n = r \); while \( BA \) will exist if \( s = m \).

6. The augmented matrix 
\[ \begin{bmatrix} 1 & 2 & 0 & -3 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & -1 & -8 \end{bmatrix} \]
represents a system of equations in the variables \( x, y, z \). Find the solution to this system of equations.

**Answer.** Perform the operations \(-R_1 + R_2 \rightarrow R_2\) and \(-R_1 + R_3 \rightarrow R_3\), and then switch \( R_2 \) and \( R_3 \) to obtain the matrix
\[ \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}. \]

Therefore, \( z = 3 \), and \( y = -5 + z = -2 \) and \( x = -3 - 2y = 1 \). Thus the solution is \((1, -2, 3)\).
7. Find the inverse of \( A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & -1 \\ -4 & -13 & 2 \end{bmatrix} \) if it exists.

**Answer.** The inverse is computed as follows

\[
\begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & -1 \\ -4 & -13 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]\n
\[
\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow -2R_1 + R_2
\]\n
\[
\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow 4R_1 + R_3
\]\n
\[
\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow -R_3
\]\n
\[
\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow R_2 + R_3
\]\n
\[
\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow -R_3 + R_1
\]\n
\[
\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow -R_3 + R_2
\]\n
\[
\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow -3R_2 + R_1
\]\n
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow -3R_2 + R_1
\]\n
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow -3R_2 + R_1
\]\n
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow -3R_2 + R_1
\]\n
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Therefore, \( A^{-1} = \begin{bmatrix} -1 & -7 & -4 \\ 0 & 2 & 1 \\ -2 & -1 & -1 \end{bmatrix} \).

8. Consider the sequence whose \( n \)th term is defined by \( a_n = (-1)^{n-1}3(2^n) \). Find \( a_1, a_2, a_3 \) and \( a_9 \).

**Answer.** \( a_1 = 3(2) = 6, \ a_2 = -3(4) = -12, \ a_3 = 3(8) = 24, \ a_9 = 3(2^9) = 1536. \)

9. Evaluate \( \sum_{k=2}^{5} \frac{(-1)^k}{2^{k+1}} \).

**Answer.** \( \sum_{k=2}^{5} \frac{(-1)^k}{2^{k+1}} = \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} = \frac{8 - 4 + 2 - 1}{64} = \frac{5}{64}. \)

10. Find the term that contains \( y^9 \) in \((2x^2 - y^3)^{11}\)

**Answer.** Let \( a = 2x^3 \) and \( b = -y^3 \). The term containing \( y^9 \) is the term containing \( b^3 \) which is

\[
\frac{11!}{3!8!}a^8b^3 = \frac{11!}{3!8!}(2x^2)^8(-y^3)^3 = 165(2^8)(x^{16})(-y^9) = -42240x^{16}y^9.
\]
11. Use the binomial theorem to expand \((x^2 - 2y)^6\).

Answer.

\[
(x^2 - 2y)^6 = \binom{6}{0}(x^2)^6(-2y)^0 + \binom{6}{1}(x^2)^5(-2y)^1 + \binom{6}{2}(x^2)^4(-2y)^2 + \binom{6}{3}(x^2)^3(-2y)^3 + \binom{6}{4}(x^2)^2(-2y)^4 + \binom{6}{5}(x^2)^1(-2y)^5 + \binom{6}{6}(x^2)^0(-2y)^6
\]

\[= x^{12} + 6x^{10}(-2y) + 15x^8(4y^2) + 20x^6(-8y^3) + 15x^4(16y^4) + 6x^2(-32y^5) + 64y^6\]

\[= x^{12} - 12x^{10}y + 60x^6y^2 - 160x^6y^3 + 240x^4y^4 - 192x^2y^5 + 64y^6\]

12. Represent \(\frac{8}{27} - \frac{16}{81} + \frac{32}{243} - \frac{64}{729}\) in summation notation.

Answer. \(\sum_{k=3}^{6}(-1)^{k+1}\frac{2^k}{3^k}\).

13. Find the equation of the circle whose graph passes through the points (5, 3), (-1, -5), and (-2, 2). (Hint: use the equation \(x^2 + y^2 + ax + by + c = 0\).)

Answer. (Hint) Plug the points in to get equations: \(5a + 3b + c = -34\), \(-a - 5b + c = -26\), and \(-2a + 2b + c = -8\). Solve by elimination to get \(a = -4\), \(b = 2\) and \(c = -20\). Therefore, the equation of the circle is 

\[x^2 + y^2 - 4a + 2b - 20 = 0\]

14. A goldsmith has two gold alloys. The first alloy is 40% gold; the second alloy is 60% gold. How many grams of each should be mixed to produce 20 grams of an alloy that is 52% gold.

Answer. Let \(x\) be the number of grams of 40% alloy and let \(y\) be the number of grams of 60% alloy. Then

\[x + y = 20 \quad \text{and} \quad .4x + .6y = .52(20)\]

Therefore, \(.4(20 - y) + .6y = 10.4\), and so \(.2y = 2.4\) and so \(y = 12\) and \(x = 8\). Thus we need 8 grams of 20% alloy and 12 grams of 60% alloy.

15. Solve the system of equations

\[
\begin{align*}
2x & - y - z = -1 \\
-x & + 3y - z = -3 \\
-5x & + 5y + z = -1
\end{align*}
\]

Answer. Add two times the second equation to the first to get \(5y - 3z = -7\) and subtract 5 times the second equation from the third to get \(-10y + 6z = 14\). These two equations are dependent and imply \(5y - 3z = -7\). Therefore, \(z = c\) where \(c\) is any number, and \(y = \frac{-7+3z}{5} = \frac{3}{5}c - \frac{7}{5}\). Now plug these back into the second equation to get \(x = 3y - z + 3 = \frac{4}{5}c + \frac{6}{5}\).

Therefore, the solution is

\((\frac{4}{5}c - \frac{6}{5}, \frac{3}{5}c - \frac{7}{5}, c)\)