Name: Hints and Answers

Instructions. Do each of the following questions. The first five question are worth 2 points each, and the last eight questions are worth 5 points each. You must show all appropriate work to obtain full credit. Good Luck!

1. Find the system of equations that is needed to find $a$, $b$ and $c$ in the equation of the the parabola $y = ax^2 + bx + c$ whose graph passes through the points $(2, 4)$, $(0, -4)$ and $(-2, 4)$. Do not solve the system of equations you found.

Answer. The three equations are:

$$4a + 2b + c = 4, \quad c = -4, \quad 4a - 2b + c = 4$$

2. The augmented matrix

$$\begin{bmatrix}
1 & 0 & 1 & -3 \\
0 & 1 & -1 & 0 \\
0 & 0 & k & -2
\end{bmatrix}$$

represents a system of equations in the variables $x$, $y$, $z$. For which value(s) of $k$ is there no solution?

Answer. $k = 0$; because in this case the third equation means $0 = -2$ which is impossible to solve.

3. The augmented matrix

$$\begin{bmatrix}
1 & 0 & 1 & -3 \\
0 & 1 & -1 & 0 \\
0 & 0 & k & -2
\end{bmatrix}$$

represents a system of equations in the variables $x$, $y$, $z$. For which value(s) of $k$ is there exactly one solution?

Answer. $k \neq 0$; this is because in this case $kz = -2$, and can be solved uniquely by dividing both sides by $k$. For example, if $k = 1$, then $z = -2$, $y = -2$, $x = -1$; if $k = 2$, then $z = -2/2 = -1$, $y = -1$ and $x = -2$, etc (you don’t want to test every possible $k$).

4. Find the 17th term of the sequence whose $n$th term is given by $a_n = (-1)^n + 3$.

Answer. $a_{17} = (-1)^{17} + 3 = -1 + 3 = 2$.

5. Let $A = \begin{bmatrix}
12 & -1 & 0 \\
-5 & 1 & 1 \\
2 & -1 & 2
\end{bmatrix}$ and $B = \begin{bmatrix}
2 & 0 & 1 & -1 & 0 \\
4 & 0 & 2 & -2 & 0 \\
-2 & 4 & -1 & 2 & -2
\end{bmatrix}$, and let $C = AB$. Find $c_{43}$ (this is the only entry of $C$ you need to find, do not find the others).

Answer. $c_{43} = \begin{bmatrix}
-2 & -1 & 3 \\
2 & -1 & 3
\end{bmatrix} = (-2)(1) + (1)(2) + (3)(-1) = -3$. 
6. A canoeist can row 12 miles with the current in 2 hours. Rowing against the current, it takes the canoeist 4 hours to travel the same distance. Find the rate of the canoeist in calm water and the rate of the current.

Answer. Let $x$ be the rate of the canoeist and $y$ be the rate of the current. Using the formula $d = rt$ we get that

$$(x + y) \cdot 2 = 12 \quad \text{and} \quad (x - y) \cdot 4 = 12.$$ 

Therefore $x + y = 6$ and $x - y = 3$. Adding these equations yields $2x = 9$, or $x = 4.5$ and then $y = 1.5$. This means the rate of the canoeist is 4.5 miles per hour, and the rate of the current is 1.5 miles per hour.

7. Find the sum $\sum_{k=2}^{4} \frac{(-1)^k}{3^{k-1}}$. Write your answer as a simplified fraction.

Answer.

$$\sum_{k=2}^{4} \frac{(-1)^k}{3^{k-1}} = \frac{(-1)^2 \cdot 2}{3} + \frac{(-1)^3 \cdot 3}{3^2} + \frac{(-1)^4 \cdot 4}{3^3}$$

$$= \frac{2}{3} - \frac{1}{3} + \frac{4}{27} = \frac{18 - 9 + 4}{27} = \frac{13}{27}.$$ 

8. Let $A = \begin{bmatrix} 0 & -3 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -1 & 4 \\ 3 & 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & -1 & 4 \\ 0 & -5 & -3 \end{bmatrix}$. Find the following, if possible, if an operation is not possible, state why it is not possible.

(a) $AB$  (b) $BA$  (c) $2B - C$

Answer. (a)

$$AB = \begin{bmatrix} 0 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -9 & -3 \\ -11 & 4 \end{bmatrix}.$$ 

(b) Not possible because the number of columns of $B$ is not equal to the number of rows of $A$.

(c) $2B - C = \begin{bmatrix} -4 & -2 & 8 \\ 6 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 4 \\ 0 & -5 & -3 \end{bmatrix} = \begin{bmatrix} -5 & -1 & 4 \\ 6 & 7 & 3 \end{bmatrix}$. 

9. (a) Find the system of equations that is equivalent to \[
\begin{bmatrix}
2 & 7 \\
1 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
2 \\
\end{bmatrix}.
\] (Do not solve the system)

Answer. \[
\begin{aligned}
2x + 7y &= 3 \\
x + 4y &= 2
\end{aligned}
\]

(b) Solve the system of equations in (a) given that the inverse of \[
\begin{bmatrix}
2 & 7 \\
1 & 4 \\
\end{bmatrix}
\]
is \[
\begin{bmatrix}
4 & -7 \\
-1 & 2 \\
\end{bmatrix}.
\]

Answer. \[
\begin{aligned}
\begin{bmatrix}
x \\
y \\
\end{bmatrix}
= 
\begin{bmatrix}
4 & -7 \\
-1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
3 \\
2 \\
\end{bmatrix}
= 
\begin{bmatrix}
12 - 14 \\
-3 + 4 \\
\end{bmatrix}
= 
\begin{bmatrix}
-2 \\
1 \\
\end{bmatrix}.
\end{aligned}
\]
Thus \(x = -2, y = 1\).

10. Use the binomial theorem or Pascal’s triangle to expand \((x^2 - 2y)^5\) (write down Pascal’s triangle if and as far as you use it).

Answer. \[
\begin{aligned}
(x^2 - 2y)^5
= & \quad (x^2)^5 + \frac{5!}{1!4!}(x^2)^4(-2y) + \frac{5!}{2!3!}(x^2)^3(-2y)^2 + \frac{5!}{3!2!}(x^2)^2(-2y)^3 + \\
& \quad + \frac{5!}{4!1!}(x^2)^1(-2y)^4 + (-2y)^5 \\
= & \quad x^{10} + 5x^8(-2y) + 10x^6(4y^2) + 10x^4(-8y^3) + 5x^2(16y^4) + (-32y^5) \\
= & \quad x^{10} - 10x^8y + 40x^6y^2 - 80x^4y^3 + 80x^2y^4 - 32y^5.
\end{aligned}
\]

11. Use the binomial theorem to find the term of \((2x - y^3)^{14}\) that contains \(x^{11}\).

Answer. The term is \[
\frac{14!}{11!3!}(2x)^{11}(-y^3)^3 = (364)(2^{11}x^{11})(-y^9) = (364)(2048)x^{11}(-y^9) = -745472x^{11}y^9.
\]
12. Solve the system of equations
\[
\begin{align*}
\begin{cases}
x + 2y - 3z &= -7 \\ 2x - y + 4z &= 11 \\ 4x + 3y - 4z &= -3
\end{cases}
\end{align*}
\]

Answer.

Multiply equation (1) by $-2$ and add that to equation (2) to obtain: $-5y + 10z = 25$ (4).

Multiply equation (1) by $-4$ and add that to equation (3) to obtain: $-5y + 8z = 25$ (5).

Subtracting (5) from (4) means $2z = 0$ and so $z = 0$. Then $-5y = 25$, and so $y = -5$. Putting these values in (1) then means $x - 10 - 0 = -7$ and so $x = 3$. Therefore, the solution is $(3, -5, 0)$.

13. Find the inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & 3 & 7 \end{bmatrix}$. Show all steps.

Answer. The inverse is computed as follows

\[
\begin{align*}
\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & 3 & 7 \end{bmatrix} & \rightarrow -2R_1 + R_2 \\
& \rightarrow -3R_1 + R_3 \\
& \rightarrow -2R_3 + R_1 \\
& \rightarrow R_3 + R_2 \\
& \rightarrow -R_2 + R_1
\end{align*}
\]

Therefore, the inverse is $A^{-1} = \begin{bmatrix} 12 & -1 & -3 \\ -5 & 1 & 1 \\ -3 & 0 & 1 \end{bmatrix}$.