Note. In all questions $b > 0$ and $b \neq 1$ where $b$ is an exponential or logarithmic base.

1. Determine whether the following functions are inverse functions. If they are inverse functions, sketch the both functions together on the same coordinate axis without using a graphing calculator.
   (a) $F(x) = 2x - 5, \quad G(x) = \frac{x + 5}{2}$.

   **Answer.**
   \[
   F(G(x)) = 2(G(x)) - 5 = 2\frac{x + 5}{2} - 5 = x + 5 - 5 = x
   \]
   \[
   G(F(x)) = \frac{F(x) + 5}{2} = \frac{2x - 5 + 5}{2} = \frac{2x}{2} = x.
   \]
   Because both $G(F(x)) = x$ and $F(G(x)) = x$, $F$ and $G$ are inverse functions. The graphs, which are lines, are left to the reader. Note, though, that they are reflections of each other about the line $y = x$.

   (b) $p(x) = \frac{x - 5}{2x}, \quad q(x) = \frac{2x}{x - 5}$.

   **Answer.**
   \[
   p(q(x)) = \frac{q(x) - 5}{2x} = \frac{\frac{2x}{x - 5} - 5}{2x} = \frac{2x - 5(x - 5)}{2x(x - 5)} = \frac{-3x + 25}{2x^2 - 10x} \neq x. \text{ Therefore, } p \text{ and } q \text{ are not inverse functions.}
   \]

2. (a) Find the inverse function of $f(x) = x^5 - 3$. Verify that the function you have found is the inverse function to $f$.

   **Answer.** Let $x = y^5 - 3$ and solve for $y$ to obtain $y = (x + 3)^{\frac{1}{5}}$. Thus $f^{-1}(x) = (x + 3)^{\frac{1}{5}}$.

   Verification: $f(f^{-1}(x)) = f((x + 3)^{\frac{1}{5}}) - 3 = (x + 3) - 3 = x$, and
   \[
   f^{-1}(f(x)) = (f(x) + 3)^{\frac{1}{5}} = (x^5 - 3 + 3)^{\frac{1}{5}} = (x^5)^{\frac{1}{5}} = x.
   \]

   (b) What is the inverse function of $h(x) = \log_{\frac{1}{4}} x$? Verify that this is the inverse function, and graph $h(x)$ and its inverse on the same coordinate axis without using a graphing calculator.

   **Answer.** The inverse function is $h^{-1}(x) = \left(\frac{1}{4}\right)^x$.

   Verification follows by using the inverse property of logs and exponentials:
   \[
   h^{-1}(h(x)) = \left(\frac{1}{4}\right)^{h(x)} = \left(\frac{1}{4}\right)^{\log_{1/4} x} = x, \text{ and}
   \]
   \[
   h(h^{-1}(x)) = \log_{\frac{1}{4}} h^{-1}(x) = \log_{\frac{1}{4}} \left(\frac{1}{4}\right)^x = x
   \]

   The graphs are left for the reader, however, again, they are reflections of each other about the line $y = x$. 

1
Suppose $f$ is a function and $g$ is its inverse function, suppose also the domain of $f$ is $[3, 10]$ and the range of $f$ is $[-2, 20]$ with $f(3) = -2$ and $f(10) = 20$.

(a) Is $f$ one-to-one?

**Answer.** Yes — because $f$ has an inverse function.

(b) Is $g$ one-to-one?

**Answer.** Yes — because $g$ has an inverse function.

(c) Find the domain and range of $g$.

**Answer.** The domain of $g$ is $[-2, 20]$ and the range of $g$ is $[3, 10]$.

(d) If possible, find: $g(3), g(-2), g(10), g(20)$.

**Answer.** $g(-2) = 3$ and $g(20) = 10$, there is not enough information to find $g(3)$ and $g(10)$.

4. Solve each equation without using a calculator:

(a) $\log_3 81 = x$

**Answer.** $3^x = 81$, or $3^x = 3^4$ and so $x = 4$.

(b) $2^{3x-1} = 32$

**Answer.** $2^{3x-1} = 2^5$ and so $3x - 1 = 5$ and so $x = 2$.

(c) $\ln e^{\pi^2} = x$

**Answer.** $x = \ln e^{\pi^2} = \pi^2 \ln e = \pi^2$, thus $x = \pi^2$.

(d) $b^{3x+2} = (b^4)^{x+1}$

**Answer.** $3x + 2 = 4(x + 1)$ and so $3x + 2 = 4x + 4$, and so $x = -2$.

5. Complete this question without the use of a graphing calculator or utility.

(a) Sketch the graph of $f(x) = 3^{|x|}$.

(b) Write the equation $y = \log_3 x$ in exponential form.

(c) Sketch the graph of $f(x) = \log_3 x$, then use reflections or translations to sketch the graphs of $g(x) = -\log_3 x$, $h(x) = -\log_3(x - 2)$, $k(x) = 4 - \log_3(x - 2)$.

(d) Sketch the graph of $f(x) = \log_3 x^4$.

**Answer.** You may check your graphs using a graphing utility. Note that for (a), if $x \geq 0$, the graph looks like the graph of $y = 3^x$ and the function is even, so the rest of the graph is a reflection about the $y$-axis.
(b) \( x = 3^y \).

(c) The graph of \( f(x) \) is reflection of the graph of \( y = 3^x \) about the line \( y = x \). The graph of \( g(x) \) is a reflection of the graph of \( f(x) \). The graph of \( h(x) \) is the graph of \( g(x) \) shifted 2 units to the right. The graph of \( k(x) \) is the graph of \( h(x) \) shifted 4 units up.

(d) Note that \( f \) is an even function so its graph is symmetric about the \( y \)-axis, and for \( x > 0 \), 

\[ \log_3 x^4 = 4 \log_3 x. \]

Note: in general, if you are given the graph of \( y = \log_b x \) without being given \( b \), you should be able to graph \( y = b^x \), \( y = \log_b x^p \), \( y = \log_b (x - h) + k \), \( y = k + b^{x-h} \) etcetera when \( b, h \) and \( k \) are given. Similarly if you are given the graph of \( y = b^x \), you should be able to graph \( y = \log_b x \), \( y = \log_b x^p \), \( y = \log_b (x - h) + k \), \( y = k + b^{x-h} \) etcetera when \( p, h \) and \( k \) are given.

6. (a) Write \( \log_b \left( \frac{\sqrt[4]{x^3z^5}}{y^2} \right) \) in terms of \( \log_b x \), \( \log_b y \) and \( \log_b z \).

Answer. \( \log_b \left( \frac{\sqrt[4]{x^3z^5}}{y^2} \right) = \frac{3}{2} \log_b x + \frac{5}{2} \log_b z - 4 \log_b y. \)

(b) Write \( 4 \log_b (x^3z^3) - 2 \log_b (\sqrt[3]{z^2}) + 3 \log_b \frac{xy}{2} \) as a single logarithm.

Answer. \( 4 \log_b (x^3z^3) - 2 \log_b (\sqrt[3]{z^2}) + 3 \log_b \frac{xy}{2} = \log_b \left( \frac{x^{15}y^2z^{10}}{8} \right). \)

7. (a) Change the equation \( \log_7 8 = -3 \) to exponential form.

(b) Change the exponential equation \( 2^{11} = 2048 \) to logarithmic form.

Answer. (a) \( \left( \frac{1}{2} \right)^{-3} = 8. \)

(b) \( \log_2 2048 = 11. \)

8. (a) Solve the equation \( \frac{e^x - e^{-x}}{e^x + 2e^{-x}} = \frac{1}{2}. \)

Answer. Multiply both sides of the equation by \( 2(e^x + 2e^{-x}) \) to obtain 

\[ 2e^x - 2e^{-x} = e^x + 2e^{-x}. \]

This simplifies to \( e^x = 4e^{-x} \). Multiply both side of this by \( e^x \) to obtain \( e^{2x} = 4. \) Thus \( 2x = \ln 4 \) and so \( x = \frac{1}{2} \ln 4 = .693147. \)

(b) Solve the equation \( 2 + \log(3x - 1) = \log(2x + 1). \)

Answer. First \( \log(2x + 1) - \log(3x - 1) = 2 \), and so \( \log \left( \frac{2x + 1}{3x - 1} \right) = 2. \) Rewriting this in exponential form we have

\[ \frac{2x + 1}{3x - 1} = 10^2. \]
Therefore $2x + 1 = 300x - 100$, or $298x = 101$ and so $x = \frac{101}{298}$. You should plug this in and verify that it works because sometimes extraneous solutions are introduced when combining the logs.

9. (Richter Scale) The magnitude of an earthquake of intensity $I$ on the Richter scale is

$$M = \log\left(\frac{I}{I_0}\right)$$

where $I_0$ is the intensity of a zero-level earthquake.

**Answer.** (Scratch Work) Before solving any of these problems, notice that $10^M = \frac{I}{I_0}$, and so $I = 10^M I_0$ is an another form of the equation.

(a) Find the magnitude to the nearest 0.1 of the 1999 Joshua Tree earthquake that had an intensity of $I = 12,589,254 I_0$.

**Answer.** $M = \log(12,589,254) \approx 7.1$.

(b) Find the intensity of the 1999 Taiwan earthquake that measured 7.6 on the Richter scale.

**Answer.** From the scratch work formula, $I = 10^{7.6} I_0 = 39,810,717 I_0$.

(c) On March 3, an earthquake in Orange County measured 2.9 on the Richter scale. Compare the intensity of the 1999 Taiwan earthquake with this earthquake. In other words, how many times more intense was the Taiwan earthquake?

**Answer.** $\frac{M_1}{M_2} = \frac{10^{7.6} I_0}{10^{2.9} I_0} = 10^{(7.6-2.9)} = 10^{4.7} \approx 50,118$. Therefore, the 7.6 earthquake was approximately 50,118 times as intense as a 2.9 earthquake.