Math 232, Final Exam, 18 March 2003

Name: Answers and Hints

1. Solve by separation of variables. Thus, \( \int \frac{dy}{y^2} = \int (t+2)dt \), and so \( y^{-1} = \frac{t^2}{2} + 2t + C \), or

\[
y = \frac{-1}{\frac{t^2}{2} + 2t + C}.
\]

Now, \( y(0) = 1 \) implies \( C = -1 \), so the solution to the initial value problem is

\[
y = \frac{-1}{\frac{t^2}{2} + 2t - 1}.
\]

2. (a) The table for Euler’s method is as follows

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \Delta t )</th>
<th>( t_i )</th>
<th>( y_i )</th>
<th>( f(t_i, y_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.25</td>
<td>.5</td>
<td>1</td>
<td>256</td>
</tr>
<tr>
<td>1</td>
<td>.75</td>
<td>1 + .25(256) = 65</td>
<td>12,960,000</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>65 + .25(12960000) = 3,240,065</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus, \( y(1) \approx 3,240,065 \).

(b) No, Bob is not correct, the equilibrium solution \( y = 5 \) will not increase to infinity, neither will an solution that starts below \( y = 5 \) since the Uniqueness Theorem applies in this setting, and so no solution can cross the equilibrium solution \( y = 5 \).

3. (a) Write the differential equation as \( \frac{dP}{dt} = 2P \left( 1 - \frac{P}{100} \right) \). Then the equilibrium points are \( P = 0 \) and \( P = 100 \). The phase line has down arrows if \( P > 100 \), and \( P < 0 \), and up arrows for \( 0 < P < 100 \). Thus any population with a positive intial value will approach \( P = 100 \) as \( t \to \infty \).

(b) \( \frac{dP}{dt} = 2P - \frac{P^2}{50} - k \).

(c) Solve \( P^2 - 100P + 50k = 0 \), and so

\[
P = \frac{100 \pm \sqrt{10000 - 200k}}{2}.
\]

The root structure changes when \( 200k = 10000 \), or \( k = 50 \). Thus, \( k = 50 \) is the bifurcation value. If more than 50 fish are harvested each year, the population will go extinct. If \( k \), where
$k$ is less than 50 fish are harvested each year, the population will tend to $\frac{100+\sqrt{10000-200k}}{2}$ fish as long as there are more than $\frac{100-\sqrt{10000-200k}}{2}$ fish to start.

4. (a) Observe that $\frac{du}{dt} = \frac{dy}{dt} - 1$, and so $\frac{dy}{dt} = \frac{du}{dt} + 1$. Thus the differential equation becomes

$$\frac{du}{dt} + 1 = u^2 - 3u - 3,$$

or

$$\frac{du}{dt} = u^2 - 3u - 4 = (u - 4)(u + 1).$$

In the variable $u$, the phaseline has equilibrium points $u = 4$ and $u = -1$, with the arrows going up when $u > 4$ and $u < -1$ and the arrows going down when $-1 < u < 4$. Thus $u = 4$ is a source and $u = -1$ is a sink.

(b) Transforming the equilibrium solutions to $y$, we get $y = t + 4$, and $y = t - 1$. Solutions starting above $y = t + 4$ will tend to infinity as $t$ increases, and tend to $y = t - 1$ going backward in time. Solutions between $y = t - 1$ and $y = t + 4$ will tend to $y = t - 1$ as $t$ increases, and will tend to $y = t + 4$ as $t$ decreases. Solutions starting below $y = t - 1$ will tend to $y = t + 1$ as $t$ increases.

(c) A description of the solutions through the given points is as follows. (i) if $y(0) = 0$, then $y(t)$ approaches the line $y = t - 1$ as $t \rightarrow \infty$ and it approaches the line $y = t + 4$ as $t \rightarrow -\infty$.

If $y(3) = 0$, then $y(t)$ approaches the line $y = t - 1$ as $t \rightarrow \infty$, and $y(t) \rightarrow -\infty$ as $t$ decreases.

(iii) if $y(0) = 4$, then $y(t) = t + 4$ is a straight-line solution.

5. (a) To pair the eigenvalues and eigenvectors we perform the multiplication:

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Therefore, 0 is paired with $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Similarly,

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = -4 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore, $-4$ is paired with $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The general solution of the system of equation is thus

$$Y(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t}.$$ 

(b) Every point along the line $y = 3x$ (corresponding to $\lambda = 0$) is an equilibrium point. Solutions starting at any point in the plane (off of the line $y = 3x$, head to the line $y = 3x$ parallel to the line $y = -x$ (which is the line corresponding to $\lambda = -3$).

(c) A solution starting at the point $(4, 0)$ will head toward the line $y = 3x$ parallel to the line $y = -x$, hence along the line $y = 4 - x$. The intersection of $y = 4 - x$ and $y = 3x$ is found by $4 - x = 3x$ and so $x = 1$, and $y = 3$. Thus a solution starting at $(4, 0)$ will head toward, but never reach the point $(1, 3)$. 
A second method for doing this is to find the solution through the point (4, 0) using the general solution in (a). One can check that

\[ Y(t) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t} \]

is the solution through that point. As \( t \to \infty \), we see that \( Y(t) \to (1, 3) \) since \( e^{-4t} \to 0 \).

6. To find the eigenvalues we solve

\[
\begin{vmatrix}
4 - \lambda & 2 \\
1 & 3 - \lambda
\end{vmatrix} = (4 - \lambda)(3 - \lambda) = \lambda^2 - 7\lambda + 10 = 0
\]

Therefore, \((\lambda - 2)(\lambda - 5) = 0\), and so \( \lambda = 2 \) and \( \lambda = 5 \). Now we find eigenvectors corresponding to these eigenvalues. For \( \lambda = 2 \), we solve

\[
\begin{bmatrix}
2 & 2 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

Therefore \( v_2 = -v_1 \), and so \( V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \). For \( \lambda = 5 \) we solve

\[
\begin{bmatrix}
-1 & 2 \\
1 & -2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

Therefore \( v_1 = 2v_2 \), and so \( V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).

(b) Using the information from (a), the general solution is

\[ Y(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}. \]

In order to find the solution through \((4, -1)\), one solves \( c_1 + 2c_2 = 4 \) and \(-c_1 + c_2 = -1\), and finds that \( c_1 = 2 \) and \( c_2 = 1 \). Thus the solution passing through \((4, -1)\) is

\[ Y(t) = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}. \]

If you prefer component form, the solution is

\[ x(t) = 2e^{2t} + 2e^{5t} \quad \text{and} \quad y(t) = -2e^{2t} + e^{5t}. \]

7. (a) The roots of \( m^2 + 6m + 8 = (m + 2)(m + 4) = 0 \) are \( m = -2 \) and \( m = -4 \). Therefore, \( y_c = c_1 e^{-2t} + c_2 e^{-4t} \) is the solution of the associated homogeneous equation. Now we guess \( y_p = Ae^{-3t} \), and so \( y'_p = -3Ae^{-3t} \) and \( y''_p = 9Ae^{-3t} \). Plugging these into the original differential equation, we get

\[(9A - 18A + 8A)e^{-3t} = 2e^{-3t}.\]
Therefore, $A = -2$, and so $y = y_c + y_p = c_1 e^{-2t} + c_2 e^{-4t} - 2e^{-3t}$ is the general solution.

(b) The particular solution will have the form

$$y_p = t(A \cos(2t) + B \sin(2t)) + t^2(C \cos(2t) + D \sin(2t)).$$

8. (a) This is a nonlinear system, the terms $-x^2$, $-3xy$ in $\frac{dx}{dt}$ are nonlinear, as are the terms $-2xy$ and $-y^2$ in $\frac{dy}{dt}$.

(b) This is a competing species system because the term $-3xy$ in $\frac{dx}{dt}$ and $-2xy$ in $\frac{dy}{dt}$ means the presence of the other species is detrimental to each species population growth. In a predator/prey model, one of these would have a positive coefficient indicating that the prey is beneficial to the population growth of the predator.

(c) We solve $x(-x - 3y + 150) = 0$ and $y(-2x - y + 100) = 0$. For the first equation, we can have $x = 0$ or $x + 3y = 150$. In the case $x = 0$, the other equation becomes $y(-y + 100) = 0$, in which case $y = 0$ or $y = 100$. Thus

$$(0, 0) \quad (0, 100)$$

are equilibrium points. In the event $x + 3y = 150$, we get $x = 150 - 3y$, and so the second equation becomes

$$y(5y - 200) = 0$$

and so $y = 0$ or $y = 40$. Hence we get the equilibrium points

$$(150, 0) \quad (30, 40)$$

Thus, in all, the system has the following four equilibrium points:

$$(0, 0), \ (0, 100), \ (150, 0), \ (30, 40).$$

9. (a) The nullclines are the lines $y = 0$, $x = 0$, $x + 3y = 150$ and $2x + y = 100$.

(b) Hint: Use the direction arrows on or through the nullclines to help you with this.

10. (a) The linearized system at $(0, 0)$ is

$$\frac{dY}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{bmatrix} Y.$$

(b) The eigenvalues for the linearized system are $\lambda = \pm i \sqrt{\frac{g}{\ell}}$, and so the equilibrium point is a center.

(c) The period is $\frac{2\pi}{\sqrt{\frac{g}{\ell}}}$.

(d) Solve $\frac{2\pi}{\sqrt{\frac{g}{\ell}}} = 1$ for $\ell$ to obtain $\ell = \frac{9.8}{4\pi^2}$ when $g = 9.8$. 