1. First, find \( y_c \) the solution to the complementary equation. For this, \( m^2 + 6m + 8 = 0 \) and so \((m+4)(m+2) = 0\). Therefore, \( m = -4, -2 \) and so \( y_c = c_1 e^{-4t} + c_2 e^{-2t} \). Now the particular solution has the form \( y_p = Ae^{-3t} \). Therefore, \( y'_p = -3Ae^{-3t} \) and \( y'' = 9Ae^{-3t} \). Plugging this back into the differential equation yields:

\[
(9A - 18A + 8A)e^{-3t} = 2e^{-3t}.
\]

Consequently, \( A = -2 \). Thus the general solution to the differential equation is

\[
y = y_c + y_p = c_1 e^{-4t} + c_2 e^{-2t} - 2e^{-3t}.
\]

Finally, to solve the initial valued problem, \( y(0) = 0 \) implies \( c_1 + c_2 - 2 = 0 \) and \( y'(0) = -2 \) implies \(-4c_1 -2c_2 + 6 = -2 \). Solving these yields \( c_1 = 2 \) and \( c_2 = 0 \). Therefore, \( y = 2e^{-4t} - 2e^{-3t} \).

2. (a) This uses the method of undetermined coefficients. Writing the equation in operator form, we get

\[
(D + 2)(D + 4)y = 2t^2 e^{-2t}.
\]

The annihilator of the RHS is \((D + 2)^3\). Therefore,

\[
(D + 2)^3(D + 4)y = 0
\]

and so \( y = c_1 e^{-4t} + c_2 e^{-2t} + c_3 te^{-2t} + c_4 t^2 e^{-2t} + c_5 t^3 e^{-2t} \). The first two terms are from the complementary solution \( y_c \), the remaining terms form \( y_p \), and so \( y_p = Ate^{-2t} + Bt^2 e^{-2t} + Ct^3 e^{-2t} \). The coefficients \( A, B \) and \( C \) can be found by plugging \( y_p \) back into the differential equation, but you were not asked to find them.

(b) \( 8\lambda^2 + b\lambda + 2 = 0 \). Therefore,

\[
\lambda = \frac{-b \pm \sqrt{b^2 - 64}}{16}.
\]

The system is overdamped if \( b > 8 \); critically damped if \( b = 8 \), and underdamped if \( 0 < b < 8 \). If \( b = 0 \), there is no damping, and \( b < 0 \) doesn’t make sense in the spring system model.

(c) If \( m\lambda^2 + 50 = 0 \), then \( \lambda = \pm i \sqrt{\frac{50}{m}} \). For resonance to occur, we need \( \lambda = \pm 5i \) and so \( m = 2 \). Resonance occurs because the frequency of the forcing function is the same as the natural frequency of the system. For all other values of \( m \), the two frequencies are different, so pure resonance will not occur.

3. (a) The eigenvalues are found by solving

\[
\begin{vmatrix}
-1 - \lambda & 2 \\
-1 & -1 - \lambda
\end{vmatrix} = 0.
\]

Therefore, \( \lambda^2 + 2\lambda + 3 = 0 \) and so

\[
\lambda = \frac{-2 \pm \sqrt{4 - 12}}{2} = -1 \pm i\sqrt{2}.
\]
Because the eigenvalues are complex with negative real part, this forms a spiral sink. The vector field at \((1, 0)\) is the vector \((-1, -1)\). Therefore, the spiral has clockwise rotation. Check this using HPG solver.

4. The eigenvalues are \(\lambda = -1 \pm i\sqrt{2}\) from 3(a). Using \(\lambda = -1 + i\sqrt{2}\) we solve
\[
\begin{bmatrix}
-i\sqrt{2} & 2 \\
-1 & -i\sqrt{2}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
One such eigenvector is when \(v_1 = 2\) and \(v_2 = i\sqrt{2}\). Thus the complex solution is
\[
Y(t) = \begin{bmatrix} 2 \\ i\sqrt{2} \end{bmatrix} e^{-t} \left[ \cos(\sqrt{2}t) + i \sin(\sqrt{2}t) \right] = \begin{bmatrix} 2 \cos(\sqrt{2}t) \\ -\sqrt{2} \sin(\sqrt{2}t) \end{bmatrix} e^{-t} + i \begin{bmatrix} 2 \sin(\sqrt{2}t) \\ \sqrt{2} \cos(\sqrt{2}t) \end{bmatrix} e^{-t}
\]

Therefore, the general (real) solution is
\[
Y(t) = c_1 \begin{bmatrix} 2 \cos(\sqrt{2}t) \\ -\sqrt{2} \sin(\sqrt{2}t) \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \sin(\sqrt{2}t) \\ \sqrt{2} \cos(\sqrt{2}t) \end{bmatrix} e^{-t}
\]

5. (a) Use HPG Solver to help you check your answer.

(b) Now, \((A - \lambda I) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\) and so \((A - \lambda I) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 - y_0 \\ x_0 - y_0 \end{bmatrix}\). Thus the general solution is
\[
Y(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} e^{-3t} + \begin{bmatrix} x_0 - y_0 \\ x_0 - y_0 \end{bmatrix} te^{-3t}
\]

(c) The solution through the point \((2, 1)\) is obtained by letting \(x = 2\) and \(y = 1\) in the general solution above, that is
\[
Y(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t}
\]
In component form, this is
\[
x(t) = 2e^{-3t} + te^{-3t} \quad y(t) = e^{-3t} + te^{-3t}.
\]
As with all other solutions (see your phase portrait from (a)), \((x(t), y(t)) \to (0, 0)\) as \(t \to \infty\).