Name: ____________________________

Instructions. Do 8 of the following 9 questions. Each is worth 5 points.

1. Consider the following system of differential equations modeling a predator-prey system.

\[
\frac{dx}{dt} = .5x \left(1 - \frac{x}{2000}\right) + .3xy \\
\frac{dy}{dt} = .7y \left(1 - \frac{y}{1000}\right) - 1.2xy
\]

Identify which variable is the predator and which is the prey. Do the predators have sources of food other than the prey? Explain your answers.

Answer. The predator is \(x\) because the \(.3xy\) terms indicates that interaction is beneficial to \(x\) while the \(-1.2xy\) term indicates that interaction is detrimental to \(y\) and so \(y\) is the prey. The predators have other sources of food because their population follows a logistic model with carrying capacity of 2000 in the absence of the prey \(y\).

2. Solve the differential equation \(\frac{dy}{dt} = ty^2 + 2y^2\) subject to \(y(0) = 1\).

Answer. Separate variables so \(\int \frac{dy}{y^2} = \int (t + 2)dt\) and so \(-\frac{1}{y} = \frac{t^2}{2} + 2t + C\). Now \(y(0) = 1\) implies \(C = -1\). Therefore, \(-\frac{1}{y} = \frac{t^2 + 4t - 2}{2}\). Solving for \(y\) yields \(y = -\frac{2}{t^2 + 4t - 2}\).

3. Use Euler’s method to estimate the solution to \(\frac{dy}{dt} = t - y^2\) subject to \(y(0) = 1\) at \(t = .5\) using \(\Delta t = 0.25\).

Answer.

<table>
<thead>
<tr>
<th>(i)</th>
<th>(\Delta t)</th>
<th>(t_i)</th>
<th>(y_i)</th>
<th>(f(t_i, y_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.25</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>.25</td>
<td>1</td>
<td>1 + .25(-1) = .75</td>
<td>-.3125</td>
</tr>
<tr>
<td>2</td>
<td>.5</td>
<td>.75 + (.25)(-.3125) = .671875</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Therefore, \(y(.5) \approx .672\).
4. Draw the phase line for the differential equation \( \frac{dy}{dt} = y^2 \left( 1 - \frac{y}{50} \right) \left( \frac{y}{100} - 1 \right) \). Identify the equilibrium points as sources, sinks or nodes. Finally, sketch solutions with the following initial conditions: (i) \( y(0) = 25 \), (ii) \( y(0) = 50 \), (iii) \( y(0) = 75 \), (iv) \( y(0) = 125 \).

**Answer.** The sketches are left to the reader. The equilibrium points are 0 (node), 50 (source), and 100 (sink). On the phaseline, the arrows are up if \( 50 < y < 100 \), the arrows are down for \( y > 100 \), \( 0 < y < 50 \) and \( y < 0 \). The long term behavior of the solutions are as follows. (i) \( y(t) \downarrow 0 \), (ii) \( y(t) = 50 \) for all \( t \), (iii) \( y(t) \uparrow 100 \), (iv) \( y(t) \downarrow 100 \).

5. Once again, Ken and Tom were arguing about solutions to \( \frac{dy}{dt} = 5y^{4/5} \) s.t. \( y(0) = -32 \). Tom said the solution cannot cross \( y = 0 \) because of the uniqueness theorem, whereas Ken said that \( y(t) = (t - 2)^5 \) is a solution to the problem, and it does cross \( y = 0 \).

(a) Is Tom correct? Explain.  (b) Is Ken correct? Explain.

**Answer.** (a) Tom is not correct. The uniqueness theorem does not apply when \( y = 0 \) because \( \frac{\partial}{\partial y}(5y^{4/5}) = 4y^{-1/5} \) is not continuous at \( y = 0 \). Thus there is no guarantee that a solution cannot cross the equilibrium solution \( y = 0 \).

(b) Ken is correct because if \( y = (t - 2)^5 \) we check that \( \frac{dy}{dt} = 5(t - 2)^4 = 5y^{4/5} \) and \( y(0) = (0 - 2)^5 = -32 \). This solution does cross the line \( y = 0 \) as it is an odd order polynomial with range \((-\infty, \infty)\).

6. It was found that a 1000 gallon tank contained 500 gallons of water containing .25 grams of mercury per gallon. Bob began filling the tank with water containing .01 grams of mercury per gallon at a rate of 10 gallons per minute while he drained the well mixed solution at a rate of 5 gallons per minute. Write an initial valued differential equation to model \( A(t) \) the amount of mercury in the tank at time \( t \). Do not solve the initial value problem.

**Answer.**

\[
\frac{dA}{dt} = (.01)(10) - \frac{A}{500 + 5t} \cdot 5 \quad \text{i.e.} \quad \frac{dA}{dt} = .1 - \frac{5A}{500 + 5t} \quad \text{s.t.} \quad A(0) = 125
\]
7. Use the change of variable \( u = \frac{y}{1 + t} \) to convert the differential equation

\[
\frac{dy}{dt} = \frac{y}{1 + t} - \frac{y}{t} + t^2(1 + t)
\]

into the form \( \frac{du}{dt} = f(u, t) \). Describe the resulting equation as linear, separable, or neither.

**Answer.** First \( y = u(1 + t) \) and so \( \frac{dy}{dt} = (1 + t)\frac{du}{dt} + u \) and so

\[
(1 + t)\frac{du}{dt} + u = u - \frac{u(1 + t)}{t} + t^2(1 + t)
\]

which simplifies to \( \frac{du}{dt} = -\frac{u}{t} + t^2 \) which is a linear equation.

8. Find the general solution to the differential equation in 7.

**Answer.** We solve this when \( t > 0 \), the solution when \( t < 0 \) is similar. First, rewrite the d.e. as \( \frac{du}{dt} + \frac{1}{t}u = t^2 \). The integrating factor is \( \mu = e^{\int t^{-1}dt} = t \). Thus \( ut = \int t^3 dt \) and so \( u(t) = \frac{t^3}{4} + \frac{C}{t} \). Now, \( y(t) = (1 + t)u(t) \) and so we get

\[
y(t) = \frac{t^3(1 + t)}{4} + \frac{C(1 + t)}{t}.
\]

9. Consider the family of d.e.’s \( \frac{dP}{dt} = -\frac{P^2}{1000} + P - 5K \) which represent a population of fish, where \( K \geq 0 \) is the parameter. Find the bifurcation value(s) of \( K \) and draw phase lines at the bifurcation values, and for values of \( K \) on either side of the bifurcation value(s).

**Answer.** We solve \( -P^2 + 1000P - 5000K = 0 \) to find that the equilibrium points are

\[
P = \frac{1000 \pm \sqrt{1,000,000 - 20,000K}}{2}
\]

The equilibrium points change in number when \( 1,000,000 - 20,000K = 0 \), that is, when \( K = 50 \). Thus the bifurcation is \( K = 50 \).

- For \( K < 50 \) the phase line has two equilibrium points \( \frac{1000 \pm \sqrt{1,000,000 - 20,000K}}{2} \), the arrows point down if \( P > \frac{1000 + \sqrt{1,000,000 - 20,000K}}{2} \), or \( P < \frac{1000 - \sqrt{1,000,000 - 20,000K}}{2} \). The arrow points up when \( P \) is between the equilibrium points.
- If \( K = 50 \), there is one equilibrium point and the arrows point down above and below the equilibrium point.
- If \( K > 50 \), there are no equilibrium points, and the arrow points down.