Math 232, Test 2, 12 February 2005

Name:  Hints and Answers

Instructions. Do any 7 of the following 8 questions. Please show all of your work. Good Luck!

1. (4 pts) Consider the predator-prey system of equations \( \frac{dx}{dt} = -x + 10xy \) and \( \frac{dy}{dt} = 3y - xy \). Find all equilibrium points of this system.

Answer. Solve \(-x + 10xy = 0\) and \(3y - xy = 0\) and so \(x(10y - 1) = 0\) and \(y(3 - x) = 0\). In the first, if \(x = 0\), then \(y = 0\) in the second. If \(y = 1/10\) in the first, then \(x = 3\) in the second. Thus the equilibrium points are \((0, 0)\) and \((3, \frac{1}{10})\).

b) (1 pt) Which variable represents the prey, and which variable represents the predator? Explain.

Answer. The variable \(x\) is the predator because interaction is beneficial to \(x\) and detrimental to \(y\) as seen from the +10\(xy\) and −\(xy\) terms in the differential equations.

2. Consider the predator prey system as in #1 where \(x\) and \(y\) are measured in 100,000’s.
(a) (2 pts) Can both species coexist with stable populations? If so, at what populations?

Answer. The can both coexist at the equilibrium point populations, i.e. \((300,000, 10,000)\).

(b) (3 pts) Would you guess that this system has large predators and small prey, or small predators and large prey. Explain.

Answer. I would guess that the prey are large and the predators are small because interaction has a relatively large benefit to the population of \(x\) (the 10\(xy\)) term while it has a relatively small affect (the −\(xy\) term) on the prey population.
3. (a) (5 pts) Use Euler’s method for systems to estimate \((x(1), y(1))\) for

\[
\frac{dx}{dt} = y \quad \frac{dy}{dt} = -x + (1 - x^2)y
\]

with \(\Delta t = .5\) given that \(x(0) = 0\) and \(y(0) = 2\).

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**Answer:** \((x(1), y(1)) = (2.5, 2.5)\)

4. (5 pts) Find the general solution to the partially decoupled system

\[
\frac{dx}{dt} = \frac{x + 3}{2} \quad \text{and} \quad \frac{dy}{dt} = xy.
\]

**Answer.** First, solve for \(x\) by separation of variables. Thus

\[
\int \frac{dx}{x+3} = \int \frac{1}{2} dt \quad \text{and so}
\]

\[
\ln |x + 3| = \frac{1}{2} t + C. \quad \text{Thus} \quad x + 3 = k_1 e^{t/2} \quad \text{and so} \quad x(t) = -3 + k_1 e^{t/2}.
\]

Now plug the solution for \(x(t)\) into \(y\) and solve for \(y\) as by separating variables follows:

\[
\int \frac{dy}{y} = \int (-3 + k_1 e^{t/2}) dt \quad \text{and so} \quad \ln |y| = -3t + 2k_1 e^{t/2} + C_2. \quad \text{Thus} \quad y(t) = k_2 e^{-3t + 2k_1 e^{t/2}}.
\]
5. (2 pts) (a) Convert the differential equation \( \frac{d^2y}{dt^2} + 8 \frac{dy}{dt} + 16y = 0 \) to a system of first order differential equations. Write the system in matrix form.

\[ \text{Answer.} \quad \text{Let } v = \frac{dy}{dt}, \text{ then } \frac{dv}{dt} = -16y - 8v \text{ Thus we get the matrix form} \]
\[
\frac{dY}{dt} = \begin{bmatrix} 0 & 1 \\ -16 & -8 \end{bmatrix} Y \quad \text{where} \quad Y = \begin{bmatrix} y \\ v \end{bmatrix}
\]

(b) (3 pts) Find all eigenvalues for the matrix you found in (a).

\[ \text{Answer.} \quad \begin{vmatrix} -\lambda & 1 \\ -16 & -8 - \lambda \end{vmatrix} = 0 \implies \lambda^2 + 8\lambda + 16 = 0 \implies (\lambda + 4)^2 = 0 \text{ which implies there is a repeated eigenvalue } \lambda = -4. \]

6. (5 pts) A linear system \( \frac{dY}{dt} = AY \) where \( A \) is a 2 by 2 matrix is known to have solutions \( Y_1(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} \) and \( Y_2(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t. \)

(a) Are \( Y_1(t) \) and \( Y_2(t) \) linearly independent?

\[ \text{Answer.} \quad \text{Yes, because } Y_1(0) \text{ and } Y_2(0) \text{ are linearly independent vectors (neither is a multiple of the other).} \]

(b) If possible, write the general solution to the system. If not possible, explain.

\[ \text{Answer.} \quad Y(t) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t. \]

(c) Is \( x(t) = e^{-2t} \) and \( y(t) = -2e^{-2t} \) a solution to the system? Explain.

\[ \text{Answer.} \quad \text{Yes, this is the solution obtained by letting } c_1 = 1 \text{ and } c_2 = 0 \text{ in (b).} \]

(d) Is \( x(t) = e^t + e^{-2t} \) and \( y(t) = 2e^t - 2e^{-2t} \) a solution to the system? Explain.

\[ \text{Answer.} \quad \text{No, to get this as a solution we would need } c_1 = 1, \ c_2 = 1 \text{ to obtain } x(t) \text{ while we would need } c_1 = 1 \text{ and } c_2 = -2 \text{ to obtain } y(t). \text{ Which is not possible.} \]
7. Consider the system \( \frac{dY}{dt} = AY \) where \( A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \). Where \( A \) has eigenvectors \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) and \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).

(a) (2 pts) Use the eigenvectors given to find the corresponding eigenvalues.

**Answer.** \[
\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \] Thus the corresponding eigenvalue is 2.

\[
\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \] Thus the corresponding eigenvalue is 5.

(b) (3 pts) Write the general solution to this system.

**Answer.** \( Y(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}. \)

8. (5 pts) Sketch the phase plane for the system in # 7, and sketch the solution that passes through the point \((-2, 0)\)

**Answer.** The equilibrium point \((0, 0)\) is a source. All arrows point away from \((0, 0)\) and \(y = -x\) and \(y = x/2\) are the straight line solutions. The solutions leave the origin tangent to the line \(y = -x\) (corresponding to the smaller eigenvalue \(\lambda = 2\)) and they eventually become nearly parallel to the line \(y = x/2\) corresponding to the larger eigenvalue \(\lambda = 5\). The sketch is left to the reader who may check it using HPG system solver. Notice that the solution through \((-2, 0)\) leaves the origin tangent to \(y = -x\) and both \(x(t) \to -\infty\) and \(y(t) \to -\infty\) at \(t \to \infty\).