Instructions. Do any five of the first six questions, and any five of the last six questions. Please do your best, and show all appropriate details in your solutions. Thank you!

1. (a) Draw the phase line for the differential equation \( \frac{dy}{dt} = -3(y - 2)^{2/3} \).
(b) Does the uniqueness theorem apply when \( y = 2 \)? Explain.
(c) Can you guarantee that no solution will ever cross the equilibrium solution \( y = 2 \)?

2. A 400-gallon tank initially contains 200 gallons of water containing 2mg of dioxin per gallon. Suppose water containing 5mg of dioxin per gallon flows into the top of the tank at a rate of 4 gallons per minute. The water in the tank is kept well-mixed, and the well-mixed water is drained at a rate of 2 gallons per minute. How much dioxin is in the tank when the tank is full?
3. Consider the differential equation \( \frac{dP}{dt} = 2P - \frac{P^2}{50} \) that models the population of a species of fish in a pond where \( t \) is measured in years.

(a) Describe the long term behavior of the population (assuming the initial population is positive).

(b) Modify the equation to represent a model where \( k \) fish are harvested each year.

(c) Find the bifurcation value for \( k \), and draw representative phase lines for \( k \) less than, equal to, and larger than this bifurcation value.

4. Use Euler’s method for systems to estimate \((x(1), y(1))\) for

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -x + (1 - x^2)y
\end{align*}
\]

with \( \Delta t = .5 \) given that \( x(0) = 0 \) and \( y(0) = 2 \).

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Answer: \((x(1), y(1)) = \)
5. Find the general solution to the partially decoupled system \( \frac{dx}{dt} = \frac{x + 3}{2} \) and \( \frac{dy}{dt} = xy. \)

6. Find the general solution to the differential equation \( \frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 13y = -3e^{-2t}. \)
7. (a) Find the general solution to \( \frac{dY}{dt} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} Y \).

(b) Find the specific solution that satisfies \( Y(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \). Write the solution in component form.

8. Consider the system \( \frac{dY}{dt} = AY \) where \( A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \) where 0, \( \begin{bmatrix} 1 \\ -2 \end{bmatrix} \) and 5, \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) are the eigenvalues and eigenvectors of \( A \).

(a) Write the general solution to this system in component form, i.e., \( x(t) = \ldots \) and \( y(t) = \ldots \).

(b) Sketch the phase portrait for the system and the solutions through (1, 4) and (4, 0).
9. Consider the system $\frac{dY}{dt} = AY$. For $A$ as given below, find the eigenvalues for $A$, and then sketch the phase portrait. Note in both cases $A$ has complex eigenvalues, but directions are still important.

(a) $A = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & -4 \\ 5 & -2 \end{bmatrix}$
10. Consider the system \( \frac{dx}{dt} = x(100 - x - 2y) \) and \( \frac{dy}{dt} = y(150 - x - 6y) \).

(a) Find all equilibrium points.

(b) Find the Jacobian at the equilibrium point \((75, 12.5)\), and then sketch the phase portrait of the linearized system at that point.

(c) In general, when can linearization fail to accurately predict the behavior of the nonlinear system near an equilibrium point?
11. Consider the system $\frac{dx}{dt} = x(100 - x - 2y)$ and $\frac{dy}{dt} = y(150 - x - 6y)$ as in question 10.

(a) Find and sketch nullclines for this system.

(b) Sketch the phase portrait for this system in the first quadrant.

(c) Considering this as a competing species model, write a brief paragraph describing the long-term behavior of the system.
Consider the system of differential equations
\[
\frac{dx}{dt} = x(1 - x) - xy \\
\frac{dy}{dt} = y(1 - y) + xy - yz \\
\frac{dz}{dt} = z(1 - z) + yz
\]

(a) Verify that \((1, 0, 0)\) is an equilibrium point of this system.

(b) Find the Jacobian of the system at this point.

(c) The Jacobian in (b) has eigenvalues with eigenvectors: 1, \[
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
2, \begin{bmatrix}
1 \\
-3 \\
0
\end{bmatrix},
-1, \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.
\]
Use this information to sketch the phase space of the linearized system at \((1, 0, 0)\).

(d) Is it likely that a solution of the original system near \((1, 0, 0)\) will approach \((1, 0, 0)\) as \(t \to \infty\)? Explain.