Math 232, Test 2, February 16, 2007

Instructions. Do each of the following 7 questions. Please show all of your work. Good Luck!

1. The following direction field is for a system of differential equations.

(a) Sketch the solution on the direction field that passes through the point $(-1, 0)$.

(b) Sketch corresponding $x(t)$, $y(t)$ plots in the space provided for time graphs.

(c) The system has one equilibrium point. Where does it appear to be?

$$(-1, 1)$$

(d) Explain why this direction field cannot be from a system of the form $\frac{dY}{dt} = AY$ where $A$ is a 2 by 2 matrix.

It does not have an equilibrium point at $(0,0)$.
2. Consider the second-order differential equation \( \frac{d^2x}{dt^2} + 2 \frac{dx}{dt} - 3x + x^3 = 0 \). Convert this to a first order system in terms of \( x \) and \( v \) where \( v = \frac{dx}{dt} \) and then find all equilibrium points.

\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= 3x - x^3 - 2v
\end{align*}
\]

Equilibrium pts:

\[
\begin{cases}
v = 0 \\
3x - x^3 = 0 \Rightarrow x(3 - x^2) = 0
\end{cases}
\]

\( x = 0 \quad x = \pm \sqrt{3} \)

\( (0,0), \ (\sqrt{3},0), \ (-\sqrt{3},0) \)

are the equilibrium points.

3. Use Euler's method for systems to estimate \((x(.2), y(.2))\) for

\[
\begin{align*}
\frac{dx}{dt} &= x - y \\
\frac{dy}{dt} &= x^2 - y
\end{align*}
\]

with \( \Delta t = .1 \) given that \( x(0) = -1 \) and \( y(0) = 2 \).

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( m_i )</th>
<th>( n_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>2</td>
<td>-3</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>-1.3</td>
<td>-1.9</td>
<td>-3.2</td>
<td>-2.1</td>
</tr>
<tr>
<td>2</td>
<td>-1.62</td>
<td>1.879</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Answer: \((x(.2), y(.2)) = (-1.62, 1.879)\)

\[
\begin{align*}
x_{i+1} &= x_i + \Delta t \cdot m_i \\
y_{i+1} &= y_i + \Delta t \cdot n_i
\end{align*}
\]
4. Find the solution to the partially decoupled system \( \frac{dx}{dt} = x + 1 \) and \( \frac{dy}{dt} = 2xy \) that passes through the point \((0, 1)\).

\[
\frac{dx}{dt} = x + 1 \quad \Rightarrow \quad \int \frac{dx}{x+1} = \int dt \quad \Rightarrow \quad \ln |x+1| = t + C \quad \Rightarrow \quad x+1 = c_1 e^t \\
\Rightarrow \quad x = -1 + c_1 e^t
\]

\[
\frac{dy}{dt} = 2xy \quad \Rightarrow \quad \frac{dy}{dt} = 2(-1 + c_1 e^t) y \quad \Rightarrow \quad \int \frac{dy}{y} = \int -2 + 2c_1 e^t \, dt \\
\Rightarrow \quad \ln |y| = -2t + 2c_1 e^t + C \\
\Rightarrow \quad y = c_2 e^{2t}
\]

To pass through \((0, 1)\) at \(t = 0\), \(c_1 = 1\)

\[
\Rightarrow \quad c_2 e^{2 \cdot 0} = 1 \quad \Rightarrow \quad c_2 = \frac{1}{e^2}
\]

\[
\therefore \quad x(t) = -1 + e^t \quad \text{and} \quad y(t) = \frac{1}{e^2} e^{2t}
\]

5. Find the general solution to the system of equations \( \frac{dY}{dt} = \begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix} Y \). To speed up the process, note that the eigenvalues of the matrix are \(-3\) and \(-6\).

\[
\lambda = -3 \quad \begin{bmatrix} -2 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{(or multiple thereof)}
\]

\[
\lambda = -6 \quad \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{(or multiple thereof)}
\]

\[
\therefore \quad Y(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-6t}
\]
6. Consider systems of the form \( \frac{dY}{dt} =AY \) where \( A \) is a 2 by 2 matrix. Sketch the phase portraits for such systems given the eigenvalues and eigenvector lines sketched below.

![Phase portrait diagrams](image)

7. Given the matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). The trace \( T \) of \( A \) is defined as \( T = a + d \). Let \( D \) be the determinant of \( A \).

(a) Show that the eigenvalues \( \lambda \) of \( A \) satisfy the equation \( \lambda^2 - T\lambda + D = 0 \).

\[
\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (a - \lambda)(d - \lambda) - bc = 0
\]

\[
\Rightarrow \quad \lambda^2 - (a + d)\lambda + ad - bc = 0
\]

\[
\Rightarrow \quad \lambda^2 - T\lambda + D = 0
\]

(b) Use the quadratic formula to find a formula for the eigenvalues involving \( T \) and \( D \).

\[
\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}
\]