Math 232, Test 1, 29 January 2008

Name: __Hints and Answers__

Instructions. Do each of the following six questions. Please show all appropriate work in your solutions in order to obtain maximum credit. You may use a calculator.

1. Solve the differential equation \( \frac{dy}{dt} = \frac{t}{y - t^2y} \) subject to \( y(0) = 4 \).

**Answer.** This is a separable equation which we rewrite as

\[
\int y \, dy = \int \frac{t}{1 - t^2} \, dt \quad \Rightarrow \\
\frac{y^2}{2} = -\frac{1}{2} \ln|1 - t^2| + C \quad \Rightarrow \\
y^2 = -\ln|1 - t^2| + t \quad \Rightarrow \\
y = \pm \sqrt{k - \ln|1 - t^2|}
\]

When \( t = 0, y = 4 \), this implies \( k = 16 \) and that we need the positive square root. Therefore, the solution is

\[ y = \sqrt{16 - \ln(1 - t^2)}, \quad -1 < t < 1. \]

2. Use Euler’s method to approximate \( y(1.5) \) for the initial valued problem \( \frac{dy}{dt} = y^2 - t \) subject to \( y(1.3) = 2 \). Use \( \Delta t = 0.1 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( t_k )</th>
<th>( y_k )</th>
<th>( m_k )</th>
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<tr>
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<td>2</td>
<td>2.7</td>
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<tr>
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<td>2.27</td>
<td>3.7529</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>2.64529</td>
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</tr>
</tbody>
</table>

**Answer:** \( y(1.5) = 2.64529 \)
3. Consider the differential equations (i) \( \frac{dy}{dt} = 3(y + 3)^{2/3} \) and (ii) \( \frac{dy}{dt} = (y + 3)^2 \).

(a) Find equilibrium points and draw phase lines for each equation.

(b) For each equation determine whether the uniqueness theorem applies for the equilibrium solution. In particular, for each equation answer whether you can guarantee that no other solution will cross an equilibrium solution.

**Answer.** (a) The equilibrium points are \(-3\) in each case. In each case, the arrows point up both above and below the equilibrium point.

(b) For (i) we cannot guarantee the uniqueness of the solution \( y = -3 \), because \( \frac{\partial f}{\partial y} = 2(y + 3)^{-1/3} \) is not continuous at \( y = -3 \). For (i), the uniqueness theorem ensures that \( y = -3 \) is unique because \( f(y) = (y + 3)^2 \) and \( \frac{\partial f}{\partial y} = 2(y + 3) \) are continuous everywhere. Consequently, no solution will cross \( y = -3 \).

4. (a) Draw the phase line for the differential equation \( \frac{dP}{dt} = 3P^2 \left( \frac{P}{500} - 1 \right) \left( 1 - \frac{P}{1000} \right) \) that models the population growth for a type of rodent. Identify the equilibrium points as sources, sinks or nodes.

(b) For which initial populations will this population survive long term? Will the population approach a stable number, if so, what will it approach?

(c) For which initial populatons will the population go extinct?

**Answer.** (a) The equilibrium point \( P = 1000 \) is a sink; \( P = 500 \) is a source; \( P = 0 \) is a sink. The arrows on the phase line are as follows: above \( P = 1000 \) the arrow points down; between \( P = 500 \) and \( P = 1000 \) the arrow points up; between \( P = 0 \) and \( P = 500 \) the arrow points down; below \( P = 0 \) the arrow points up. Please draw it!

(b) The population will survive long term if \( P \geq 500 \). If \( P(0) = 500 \), it remains in equilibrium; if \( P(0) > 500 \), then it approaches 1000 as \( t \to +\infty \).

(c) \( P(0) < 500 \).
5. Consider the one parameter family of differential equations \( \frac{dy}{dt} = ky - y^3 \). Determine the bifurcation values for \( k \), and draw representative phase lines at, and on each side of the bifurcation value(s).

**Answer.** Write \( \frac{dy}{dt} = y(k - y^2) \). If \( k > 0 \), there are three equilibrium points \( y = 0, y = \sqrt{k}, \) \( y = -\sqrt{k} \). If \( k \leq 0 \), there is one equilibrium point \( y = 0 \). Therefore, a bifurcation occurs when \( k = 0 \).

Please draw the phase lines which are described as follows.

When \( k \leq 0 \): the arrow points up below \( y = 0 \), and the arrow points down above \( y = 0 \).

When \( k > 0 \): the arrow points down above \( y = \sqrt{k} \), the arrow points up between \( y = 0 \) and \( y = \sqrt{k} \); the arrow points down between \( y = -\sqrt{k} \) and \( y = 0 \); the arrow points up below \( y = -\sqrt{k} \).

6. An 1800 liter spa initially contains 600 liters of pure water. Water containing 3mg per liter of chlorine is pumped into the spa at a rate of 60 liters per minute, while the well-mixed water is discharged at a rate of 30 liters per minute. Find the amount (in mg) and concentration (in mg/liter) of chlorine in the spa when the spa is full?

**Answer.** Let \( A \) be the amount (mg) of chlorine in the spa, and let time be measured in minutes. The initial value problem is then

\[
\frac{dA}{dt} = 3(60) - 30 \frac{A}{600 + 30t} = 180 - \frac{A}{20 + t} \quad \text{subject to } A(0) = 0.
\]

Rewrite this as \( \frac{dA}{dt} + \frac{1}{20 + t} A = 180 \) and so the integrating factor is

\[
\mu = e^{\int \frac{1}{20+t} \, dt} = 20 + t
\]

Consequently, \( \mu A = \int 180(20 + t) \, dt \), this implies

\[
(20 + t)A = 90(20 + t)^2 + C; \quad \text{or, in other words} \quad A = 90(20 + t) + \frac{C}{20 + t}.
\]

Now, \( A(0) = 0 \) implies \( 0 = 1800 + \frac{C}{20} \) and so \( C = -36,000 \). Therefore,

\[
A(t) = 90(20 + t) - \frac{36000}{20 + t}.
\]

The spa is full when \( t = 40 \). Then \( A(40) = (90)(60) - \frac{36000}{60} = 4800 \).

Thus, there are 4800mg of chlorine in the spa when it is full, and the concentration is \( \frac{4800}{1800} \text{ liters} = \frac{8}{3} \text{ mg/liter of chlorine} \).