Instructions. Do each of the following six problems. Please do your best, and show all appropriate details in your solutions. Thank you!

1. (10 pts) (a) Find the general solution to \( \frac{dY}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} Y \).

(b) Find the solution through the point (2, 3).

2. (a) (3 pts) Write the second-order differential equation for the unforced harmonic oscillator with mass \( m = 2 \), and damping coefficient \( b = 4 \) with unknown spring constant \( k > 0 \).

(b) (3 pts) Determine values for \( k \) for which the oscillator is (i) overdamped; (ii) critically damped; (iii) underdamped.

(c) (4 pts) Write the general solution to the differential equation when \( k = 4 \).
3. (10 pts) Sketch the phase portraits for the following systems of differential equations with the help of the given information about their eigenvalues and/or eigenvectors. Directions are important!

(a) \( \frac{dY}{dt} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} Y \). Eigenvalues: repeated eigenvalue \( \lambda = 3 \), with eigenvector \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

(b) \( \frac{dY}{dt} = \begin{bmatrix} 0 & -2 \\ 2 & 1 \end{bmatrix} Y \). Eigenvalues: \( \lambda = \frac{1 \pm i \sqrt{15}}{2} \).

(c) \( \frac{dY}{dt} = \begin{bmatrix} 1 & 5 \\ -2 & -1 \end{bmatrix} Y \). Eigenvalues: \( \lambda = \pm 3i \).
4. Let $A$ be a 2 by 2 matrix.

(a) Show that the eigenvalues of $A$ satisfy the equation $\lambda^2 - \lambda T + D = 0$ where $T$ is trace of $A$ and $D$ is the determinant of $A$.

(b) Use the quadratic formula to determine when (in terms of $T$ and $D$) that $A$ has: complex eigenvalues, repeated real eigenvalue, distinct real eigenvalues.

5. Consider the differential equation $\frac{d^2y}{dt^2} + 8 \frac{dy}{dt} - 9y = f(t)$

(a) (4 pts) Find $y_h$, the solution to the associated homogeneous equation.

(b) (2 pts) Determine the form of an appropriate guess for $y_p$ if $f(t) = 4t$.

(c) (2 pts) Determine the form of an appropriate guess for $y_p$ if $f(t) = e^{3t}$.

(d) (2 pts) Suppose $f(t) = e^{kt}$; for what values of $k$ would a “second guess” for the from of $y_p$ be required?
6. Consider the forced undamped harmonic oscillator represented by \( \frac{d^2 y}{dt^2} + 25y = 5 \cos(5t) \).

(a) (2pts) Does this represent an oscillator where pure resonance occurs? Explain.

(b) (8 pts) Find the general solution to the given differential equation.