Math 233: Final Test, 6 June 2005

Name: ______________________________

Instructions: You must do problem 1, but you may skip one of problems 2 through 10. Please show all of your work, and have a great summer.

1. (20 pts) (a) Find the limit \( \lim_{(x,y) \to (-1,2)} \frac{xy}{x^2 + y} \).

(b) Describe the domain of \( f(x, y) = \sqrt{x^2 + y^2 - 16} \).

(c) Given that \( x^2 e^y + xy = 5 \) defines \( y \) as a differentiable function of \( x \). Find \( \frac{dy}{dx} \).

(d) Describe the region whose area is given by \( \int_{\pi/2}^{\pi} \int_0^4 r \, dr \, d\theta \).

(e) Find a vector normal to the surface \( x^2 + y^2 - z^2 = 0 \) at the point \( (4, -3, 5) \).

(f) Describe the solid whose volume is given by \( \int_0^{2\pi} \int_0^{\pi/4} \int_0^5 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \).

(g) If \( h(x, y) \) is the height of a wall at each point \( (x, y) \) on a smooth curve \( C \), what does the line integral \( \int_C h(x, y) \, ds \) represent?

(h) Write a formula for \( ds \) for the curve \( x = y^3 \).

(i) Parametrize the smooth surface that is the wall of the cylinder \( x^2 + y^2 = 4 \) where \( -2 \leq z \leq 8 \).

(j) Given a smooth surface \( z = g(x, y) \), write the formula for \( \mathbf{N} \, ds \) where \( \mathbf{N} \) is the upward unit normal.
2. (a) (5 pts) The centripetal acceleration of a particle moving in a circle is \( a = \frac{v^2}{r} \), where \( v \) is the velocity and \( r \) is the radius of the circle. Use differentials to approximate the maximum percent error in measuring the acceleration due to errors 3% in \( v \) and 2% in \( r \).

(b) (5 pts) Let \( F(x, y, z) = x^2y + e^z + yz \) where \( x = \frac{u}{v^3} \), \( y = 3v - u \) and \( z = e^v \). Find \( \frac{\partial F}{\partial u} \) when \( v = 1 \) and \( u = -2 \).

3. (a) (5 pts) Find the directional derivative of the function \( f(x, y) = x^2y - 2xy^2 \) in the direction \( \mathbf{v} = -5\mathbf{i} + 12\mathbf{j} \) at the point \((2, 1)\).

(b) (5 pts) Find all points on the surface \( z = 3x^2 + 2y^2 - 3x + 4y + 12 \) where the tangent plane is horizontal. Write the equations of the tangent planes at those points.
4. (10 pts) Find and classify relative extrema of \( f(x, y) = x^3 - 3xy + y^3 \).

5. (10 pts) Evaluate \( \int_0^1 \int_{3x}^3 e^{-y^2} \, dy \, dx \) by reversing the order of integration.

6. (10 pts) Find the mass of a solid with density \( \rho(x, y, z) = x^2 + y^2 \) that is below the plane \( z = 8 \) and above the cone \( z = 2\sqrt{x^2 + y^2} \).
7. (10 pts) Sketch the solid whose volume is given by the integral \( \int_{0}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{6-x-y} dz \, dy \, dx \), and rewrite the integral in the order \( dz \, dx \, dy \) (do not evaluate the integrals).

8. (a) (5 pts) Find the surface area of a wall whose height is \( h(x, y) = y \) along the top half of the circle \( x^2 + y^2 = 9 \).

(b) (5 pts) Evaluate the line integral \( \int_{C} x \, dy + y \, dx \) where \( C \) is the portion of \( x = y^2 \) that starts at \( (4, -2) \) and ends at \( (1, 1) \).
9. Let $S$ be the portion of the plane $2x + 3y + 2z = 12$ that is inside the cylinder $x^2 + y^2 = 16$, and let $\mathbf{F}(x, y, z) = 2y\mathbf{i} - 2x\mathbf{j} + z\mathbf{k}$.

(a) (5 pts) Set-up but do not evaluate the integral to find the flux of the vector field $\mathbf{F}$ through the surface $S$ oriented with upward unit normal.

(b) (5 pts) Let $C$ be the boundary of the surface $S$ as above, oriented in a counter-clockwise direction. Use Stokes’ theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

10. (10 pts) Let $S$ be the surface of the sphere $x^2 + y^2 + z^2 = 9$ oriented with outward unit normal. Let $\mathbf{F}(x, y, z) = (2x + y\cos z)\mathbf{i} + (y - z)\mathbf{j} + (2z - 3xe^{xy})\mathbf{k}$.

(a) Find the flux of $\mathbf{F}$ through the closed surface $S$.

(b) Evaluate the integral $\iint_S \text{curl}\mathbf{F} \cdot \mathbf{N}dS$. 