1. (20 pts) (a) \( \lim_{(x,y) \to (-1,2)} \frac{xy}{x^2 + y} = \frac{(-1)(2)}{(-1)^2 + 2} = -\frac{2}{3}. \)

(b) The domain of \( f(x, y) = \sqrt{x^2 + y^2 - 16} \) is \( \{(x, y) : x^2 + y^2 \geq 16\} \) which all points in the \( xy \)-plane that are on or outside the circle of radius 4 centered at the origin.

(c) Given \( x^2 e^y + xy = 5 \) we let \( F(x, y) = x^2 e^y + xy \). Then \( \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2xe^y + y}{x^2e^y + x}. \)

(d) The integral \( \int_{\pi/2}^{\pi} \int_{0}^{4} r \, dr \, d\theta \) gives the area of the region in the 2nd quadrant that is enclosed by the circle of radius 4 centered at the origin.

(e) A vector normal to the surface \( x^2 + y^2 - z^2 = 0 \) at the point \((4, -3, 5)\) can be found by using the gradient. That is, \( \nabla G(x, y, z) = (2x, 2y, -2z) \) and so \( G(4, -3, 5) = (8, -6, -10) \) is a normal vector to the surface.

(f) The integral \( \int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{0}^{5} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \) gives the volume of the solid lies above the cone \( z = \pi/4 \) and inside the sphere \( \rho = 5 \).

(g) If \( h(x, y) \) is the height of a wall at each point \((x, y)\) on a smooth curve \( C \), then the line integral \( \int_{C} h(x, y) \, ds \) represents the surface area of the wall.

(h) A formula for \( ds \) for the curve \( x = y^3 \) is \( ds = \sqrt{9t^4 + 1} \, dt \).

(i) The wall of the cylinder \( x^2 + y^2 = 4 \) where \(-2 \leq z \leq 8\) can be parametrized by \( P(\theta, z) = (2 \cos \theta, 2 \sin \theta, z) \) where \( 0 \leq \theta \leq 2\pi \) and \(-2 \leq z \leq 8\).

(j) Given a smooth surface \( z = g(x, y) \), a formula for \( \mathbf{N} \, dS \) where \( \mathbf{N} \) is the upward unit normal is \( \langle -g_x, -g_y, 1 \rangle \, dx \, dy \).
2. (a) (5 pts) The centripetal acceleration of a particle moving in a circle is \( a = \frac{v^2}{r} \), where \( v \) is the velocity and \( r \) is the radius of the circle. Use differentials to approximate the maximum percent error in measuring the acceleration due to errors 3% in \( v \) and 2% in \( r \).

**Answer.** The differential for \( a \) is

\[
\begin{align*}
\frac{da}{dv} \, dv + \frac{da}{dr} \, dr &= \frac{2v}{r} \, dv - \frac{v^2}{r^2} \, dr \\
&= \frac{2v}{r} \, (\pm 0.03v) - \frac{v^2}{r^2} \, (\pm 0.02v)
\end{align*}
\]

Thus, with appropriate choices of signs \( da = \pm 0.08 \frac{v^2}{r} \), or \( da = \pm 0.08a \) which is an error of 8%.

(b) (5 pts) Let \( F(x, y, z) = x^2y + e^z + yz \) where \( x = \frac{u}{v^3} \), \( y = 3v - u \) and \( z = e^v \). Find \( \frac{\partial F}{\partial u} \) when \( v = 1 \) and \( u = -2 \).

**Answer.** Using the chain rule,

\[
\frac{\partial F}{\partial u} = F_x \frac{\partial x}{\partial u} + F_y \frac{\partial y}{\partial u} + F_z \frac{\partial z}{\partial u}
\]

\[
= (2xy) \left( \frac{1}{v^3} \right) + (x^2 + z)(-1) + (e^z + y)(0)
\]

Evaluating this when \( u = -2 \) and \( v = 1 \), we get

\[
\frac{\partial F}{\partial u} = 2 \left( \frac{-2}{1} \right) (5)(1) + ((-2)^2 + e)(-1) = -20 - 4 - e = -24 - e.
\]

3. (a) (5 pts) Find the directional derivative of the function \( f(x, y) = x^2y - 2xy^2 \) in the direction \( \mathbf{v} = -5\mathbf{i} + 12\mathbf{j} \) at the point \((2, 1)\).

**Answer.** A unit vector in the direction of \( \mathbf{v} \) is \( \mathbf{u} = \left( -\frac{5}{13}, \frac{12}{13} \right) \). Also, \( \nabla f(x, y) = (2xy - 2y^2, x^2 - 4xy) \) and so \( \nabla f(2, 1) = (2, -4) \). Then

\[
D_\mathbf{u} f(2, 1) = (2, -4) \cdot \left( -\frac{5}{13}, \frac{12}{13} \right) = -\frac{58}{13}.
\]

(b) (5 pts) Find all points on the surface \( z = 3x^2 + 2y^2 - 3x + 4y + 12 \) where the tangent plane is horizontal. Write the equations of the tangent planes at those points.

**Answer.** The tangent plane is horizontal when \( z_x = 0 = z_y \), and so we solve \( z_x = 6x - 3 = 0 \) and \( z_y = 4y + 4 = 0 \). This leads us to the point \( (1/2, -1, 37/4) \). Thus the tangent plane is \( z = 37/4 \).
4. (10 pts) Find and classify relative extrema of \( f(x, y) = x^3 - 3xy + y^3 \).

**Answer.** First we compute the critical points: \( f_x = 3x^2 - 3y = 0 \) and \( f_y = -3x + 3y^2 = 0 \) and so \( y = x^2 \) and \(-3x + 3x^4 = 0 \) which imply \(-3x(1 - x^3) = 0 \) and so \( x = 0, 1 \). Thus the critical points are \((0, 0)\) and \((1, 1)\).

Now, \( f_{xx} = 6x, f_{xy} = -3 \) and \( f_{yy} = 6y \).

At \((0, 0)\), \( D = f_{xx}f_{yy} - f_{xy}^2 = 0 - 3^2 < 0 \), and so there is a saddle point at \((0, 0)\).

At \((1, 1)\), \( D = 6^2 - 3^2 > 0 \), and \( f_{xx} = 6 > 0 \) and so there is a local minimum at \((1, 1)\).

5. (10 pts) Evaluate \( \int_0^1 \int_{3x}^3 e^{-y^2} dy \, dx \) by reversing the order of integration.

**Answer.** The region of integration is \( 0 \leq x \leq 1 \) and \( 3x \leq y \leq 3 \) which written in the reversed order becomes \( 0 \leq y \leq 3 \) and \( 0 \leq x \leq y/3 \). Therefore,

\[
\int_0^1 \int_{3x}^3 e^{-y^2} dy \, dx = \int_0^3 \int_0^{y/3} e^{-y^2} dx \, dy = \int_0^3 \frac{y}{3} e^{-y^2} - \frac{1}{6} e^{-y^2} \bigg|_0^3 = \frac{1}{6} (1 - e^{-9}).
\]

6. (10 pts) Find the mass of a solid with density \( \rho(x, y, z) = x^2 + y^2 \) that is below the plane \( z = 8 \) and above the cone \( z = 2\sqrt{x^2 + y^2} \).

**Answer.** The mass of the solid is given in cylindrical coordinates by

\[
\int_0^{2\pi} \int_0^4 \int_0^8 r^3 dr \, d\theta \, dz = \int_0^{2\pi} \int_0^4 (8r^3 - 2r^4) d\theta \, dr \, dz = 2\pi \left( 2r^4 - \frac{2}{5} r^5 \right) \bigg|_0^4 = 2\pi \frac{512}{5} = \frac{1024\pi}{5}.
\]
7. (10 pts) Sketch the solid whose volume is given by the integral \( \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{6-x-y} dz 
 dy 
 dx \), and rewrite the integral in the order \( dz 
 dx 
 dy \) (do not evaluate the integrals).

**Answer.** The sketch is left to the reader. It is the solid that lies below the plane \( x + y + z = 6 \) and above \( z = 0 \) and it projects onto the \( xy \)-plane as a quarter circle of radius 3 in the first quadrant. In otherwords, it is the solid in the first octant that is below the plane \( x + y + z = 6 \) and inside the cylinder \( x^2 + y^2 = 9 \).

In the new order of integration, the integral is

\[
\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{6-x-y} dz 
 dy 
 dx
\]

8. (a) (5 pts) Find the surface area of a wall whose height is \( h(x, y) = y \) along the top half of the circle \( x^2 + y^2 = 9 \).

**Answer.** The parametrize the top half of the circle as \( x = 3 \cos t, \ y = 3 \sin t \) where \( 0 \leq t \leq 2\pi \) and then \( ds = \sqrt{9 \sin^2 t + 9 \cos^2 t} \ dt = 3 \ dt \). Thus, the area is given by the line integral

\[
\int_0^\pi (3 \sin t)(3 \ dt) = 18.
\]

(b) (5 pts) Evaluate the line integral \( \int_C x \ dy + y \ dx \) where \( C \) is the portion of \( x = y^2 \) that starts at \( (4, -2) \) and ends at \( (1, 1) \).

**Answer.** Let \( x = t^2 \) and \( y = t \) where \(-2 \leq t \leq 1\). Thus

\[
\int_C x \ dy + y \ dx = \int_{-2}^1 t^2 \ dt + t(2t \ dt) = \int_{-2}^1 3t^2 \ dt = t^3 \bigg|_{-2}^1 = 9.
\]
9. Let \( S \) be the portion of the plane \( 2x + 3y + 2z = 12 \) that is inside the cylinder \( x^2 + y^2 = 16 \), and let \( \mathbf{F}(x, y, z) = 2y\mathbf{i} - 2x\mathbf{j} + z\mathbf{k} \).

(a) (5 pts) Set-up but do not evaluate the integral to find the flux of the vector field \( \mathbf{F} \) through the surface \( S \) oriented with upward unit normal.

Answer. \( N\mathbf{d}S = \langle 1, \frac{3}{2}, 1 \rangle dA \) and so we get the flux integral as

\[
\int \int_S \langle 2y, -2x, z \rangle \cdot \langle 1, 3/2, 1 \rangle \, dA = \int \int_R (2y - 3x + 6 - x - 3y/2) \, dA
\]
\[
= \int_0^{2\pi} \int_0^4 (6 - 4r \cos \theta + \frac{1}{2} r \sin \theta) \, r \, dr \, d\theta.
\]

(b) (5 pts) Let \( C \) be the boundary of the surface \( S \) as above, oriented in a counter-clockwise direction. Use Stokes’ theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \).

Answer. Compute that \( \text{curl} \mathbf{F} = \langle 0, 0, -4 \rangle \), and \( |\mathbf{b}f N\mathbf{d}S = \langle 1, 3/2, 1 \rangle \, dA \) as above. Now

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_S \text{curl} \mathbf{F} \cdot N \, dS = \int \int_R -4 \, dA = (-4)(16\pi) = -64\pi.
\]

10. (10 pts) Let \( S \) be the surface of the sphere \( x^2 + y^2 + z^2 = 9 \) oriented with outward unit normal. Let \( \mathbf{F}(x, y, z) = (2x + y\cos z)\mathbf{i} + (y - z)\mathbf{j} + (2z - 3xe^{xy})\mathbf{k} \).

(a) Find the flux of \( \mathbf{F} \) through the closed surface \( S \).

Answer. We’ll use the divergence theorem. First, \( \text{div} \mathbf{F} = 2 + 1 + 2 = 5 \) and let \( Q \) denote the ball of radius 3 enclosed by \( S \). Now

\[
\int \int \mathbf{F} \cdot N\mathbf{d}S = \int \int \int_Q 5\, dV = 5(\text{vol}Q) = 5 \cdot \frac{4}{3}\pi 3^3 = 180\pi.
\]

(b) Evaluate the integral \( \int \int_S \text{curl} \mathbf{F} \cdot N\mathbf{d}S \).

Answer. In general, \( \text{div}(\text{curl} \mathbf{F}) = 0 \) when \( \mathbf{F} \) has continuous 2nd partial derivatives, thus the divergence theorem says

\[
\int \int_S \text{curl} \mathbf{F} \cdot N\mathbf{d}S = \int \int \int_Q 0\, dV = 0.
\]