Instructions. Do question 1 and any four of questions 2 through 6. Please justify all answers.

1. (a) (2 pts) Define what is meant by a conservative vector field \( \mathbf{F} \).

(b) (3 pts) Given a vector field \( \mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k} \) write the formulas for \( \text{div}\mathbf{F} \) and \( \text{curl}\mathbf{F} \).

(c) (2 pts) If \( \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \) where \( C \) is a closed, simple, piecewise smooth curve, is \( \mathbf{F} \) conservative? Explain.

(d) (4 pts) Sketch some representative vectors for the vector field \( \mathbf{F}(x, y) = -yi + xj \).

(e) (2 pts) Explain in words how the sketch of the vector field \( \mathbf{G}(x, y) = yi - xj \) compares to that in (d).

(f) (2 pts) With the help of Green’s theorem write a formula \( \int_C M\,dx + N\,dy \) when \( C \) is a simple piecewise smooth curve enclosing a region \( R \) and \( C \) is oriented in a clockwise direction, and assume \( M \) and \( N \) have continuous partial derivatives.

2. (10 pts) A retaining wall has its base along the curve \( y^3 = x \) starting at \((-8, -2)\) and ending at \((1, 1)\). The height of the wall is \( h(x, y) = x^2y^2 \). Completely set-up (but do not evaluate) a line integral to find the surface area of the retaining wall.

3. (10 pts) Find the work done by the force field \( \mathbf{F}(x, y, z) = -\frac{1}{2}xi - \frac{1}{2}yj + \frac{1}{4}k \) on a particle as it moves along the helix given by \( \mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + tk \) from the point \((1, 0, 0)\) to \((-1, 0, 3\pi)\).

4. (10 pts) Tom computed that the integral \( \int_C y^2\,dx \) is 0 where \( C \) is the unit circle oriented in a counter clockwise direction. He then concluded that \( \mathbf{F}(x, y) = y^2\mathbf{i} \) is a conservative vector field because it is path independent. Is Tom correct? Explain. (Address both the value of the line integral, and his reasoning about \( \mathbf{F} \) being conservative).

5. (10 pts) (a) Use Green’s theorem to show that the area of a plane region \( R \) bounded by a piecewise smooth simple closed curve \( C \), oriented counter clockwise, is given by \( A = \frac{1}{2} \int_C x\,dy - y\,dx \).

(b) Use the formula in (a) to find the area of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

6. (10 pts) Verify that the vector field \( \mathbf{F}(x, y) = (2xy + 1)\mathbf{i} + (x^2 + ye^y \cos y)\mathbf{j} \), is conservative, and then evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is the path consisting of line segments from \((0, 0)\) to \((1, 1)\) and then from \((1, 1)\) to \((4, 10)\) and then from \((4, 10)\) to \((5, 0)\).