Math 233: Final Test, 14 June 2006

Name: ____________________________

Instructions: You must do problem 1, but you may skip one of problems 2 through 10. Please show all of your work, and have a great summer.

1. (20 pts) (a) Describe the level curve of \( f(x, y) = x^2 + y^2 \) at level \( c = 25 \).

(b) Given that \( y^2 \sin x - xy^3 = 5 \) defines \( y \) as a differentiable function of \( x \). Find \( \frac{dy}{dx} \).

(c) Find a vector normal to the level curve \( \frac{x^2}{4} + \frac{y^2}{9} = 2 \) at the point \((-2,3)\).

(d) Sketch the region of integration for \( \int_0^3 \int_0^{2y} f(x, y) \, dx \, dy \).

(e) Write the double integral from (d) using the reversed order \( dy \, dx \).

(f) Describe the solid whose volume is given by \( \int_0^\pi \int_0^2 \int_0^5 r \, dz \, dr \, d\theta \) in cylindrical coordinates.

(g) Parametrize the line segment from \((5,0)\) to \((2,15)\).

(h) Using the fundamental theorem of line integrals and potential functions, explain why a conservative vector field is path independent.

(i) Parametrize the smooth surface given by \( z = x^2 + y^2 \) where \( z \leq 9 \).
2. (a) (5 pts) Let \( w = xy + \ln z \) where \( x = \frac{v^2}{u} \), \( y = u + v \) and \( z = \cos u \). Find \( \frac{\partial w}{\partial v} \) when \( u = -1 \) and \( v = 2 \).

(b) (5 pts) Calculate the directional derivative of the function \( f(x, y) = 2xy - y^3 \) in the direction \( \langle 3, -4 \rangle \) at the point \( (1, 3) \).

3. (a) (5 pts) Write an equation for a plane through the point \( (x_0, y_0, z_0) \) with normal vector \( n = \langle a, b, c \rangle \). Then with the help of a picture, use the dot product to explain why your equation is correct.

(b) (5 pts) Find an equation of a tangent plane to the surface \( x^2 + y^2 - z^2 = 0 \) at the point \( (-3, 4, -5) \).
4. (10 pts) Find the point where the function $f(x, y, z) = 3x^2 + y^2 + 2z^2$ attains its minimum when restricted to the plane $2x + y - z = 17$.

5. (10 pts) Evaluate the integral $\int_0^2 \int_{y/2}^1 e^{-x^2} \, dx \, dy$ by reversing the order of integration.
6. (10 pts) Consider the planar lamina bounded by $x = 16 - y^2$ and $x = -9$ whose density is given by $\rho = kx^2y^2$.

(a) Write the formulas for $\bar{x}$ and $\bar{y}$ in terms of $M, M_x$ and $M_y$.

(b) Write the integrals needed to find $M, M_x$ and $M_y$ for this lamina, but do not evaluate the integrals.

7. (10 pts) Sketch the solid whose volume is given by $\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{-\sqrt{x^2-y^2}}^{x^2-y^2} dz \ dy \ dx$ and evaluate the integral by converting it to either cylindrical or spherical coordinates (whichever looks easier).
8. (a) (5 pts) Find the surface area of a wall whose height is \( h(x, y) = x \) along the circle \( x^2 + y^2 = 1 \) where \( x \geq 0 \).

(b) (5 pts) Evaluate the line integral \( \int_C x^2 \, dy - y \, dx \) where \( C \) is the portion of \( x = y^2 \) that starts at \((4, -2)\) and ends at \((1, 1)\).

9. Let \( S \) be the portion of the plane \( 2x + 3y + 2z = 12 \) that is inside the cylinder \( x^2 + y^2 = 16 \).

(a) (5 pts) Set-up but do not evaluate a surface integral to find the mass of the surface \( f \) given that its density per unit area is \( \rho(x, y, z) = x^2 + y^2 + z^2 \).

(b) (5 pts) Set up but do not evaluate a surface integral to find the flux of the vector field \( \mathbf{F} = 2x \mathbf{i} - xy \mathbf{j} + 5k \) through the surface \( S \) oriented with upward unit normal.
10. Let $S$ be the surface of the rectangular box $-1 \leq x \leq 2$, $0 \leq y \leq 3$ and $0 \leq z \leq 5$ oriented with outward unit normal. Let $\mathbf{F}(x, y, z) = 4x\mathbf{i} + x\mathbf{j} + (2z - 1)\mathbf{k}$.

(a) (5 pts) Find the flux of $\mathbf{F}$ through the closed surface $S$.

(b) (5 pts) Use Stokes’ theorem evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ as a double integral, where $\mathbf{F}$ is as above, and $C$ is the boundary of the top side of the box described above oriented in a counter clockwise direction when viewed from above.

Extra Credit. On the back of this sheet, derive the equation for the “least squares” line of best fit to the data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. 