Math 233: Final Test, 6 June 2005

Name: _______ 

Instructions: You must do problem 1, but you may skip one of problems 2 through 10. Please show all of your work, and have a great summer.

1. (20 pts) (a) Describe the level curve of \( f(x, y) = x^2 + y^2 \) at level \( c = 25 \).

Circle, radius 5 centered at origin.

(b) Given that \( y^2 \sin x - xy^3 = 5 \) defines \( y \) as a differentiable function of \( x \). Find \( \frac{dy}{dx} \).

\[
\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-y^2 \cos x - y^3}{2y \sin x - 3xy^2} = \frac{y^2 - y^2 \cos x}{2y \sin x - 3xy^2} = \frac{y^2 - y^2 \cos x}{2y \sin x - 3xy^2}
\]

(c) Find a vector normal to the level curve \( \frac{x^2}{4} + \frac{y^2}{9} = 2 \) at the point \((-2, 3)\).

\( \nabla f = \langle \frac{x}{2}, \frac{y}{3} \rangle \)

\( \nabla f (-2, 3) = \langle \frac{-2}{2}, \frac{3}{9} \rangle = \langle -1, \frac{1}{3} \rangle \)

(d) Sketch the region of integration for \( \int_0^3 \int_0^{2y} f(x, y) \, dx \, dy \).

(e) Write the double integral from (d) using the reversed order of integration.

\[
\int_0^2 \int_0^3 f(x, y) \, dy \, dx
\]

(f) Describe the solid whose volume is given by \( \int_0^\pi \int_0^2 \int_0^5 r \, dz \, dr \, d\theta \).

Half a cylinder of radius 2, height 5.

(Th half whose \( y \geq 0 \) and \( 0 \leq z \leq 5 \).

The right circular cylinder is centered on \( x = 1 \).

(g) Parametrize the line segment from \((5, 0)\) to \((2, 15)\).

\[
x = 5 - 3t, \quad 0 \leq t \leq 1
\]

\[
y = 15t
\]

(h) Using the fundamental theorem of line integrals and potential functions, explain why a conservative vector field is path independent.

The integral \( \int \vec{F} \cdot d\vec{s} \) (work done by \( \vec{F} \)) is the difference of the potential function at the endpoints. It depends only on endpoints, not on path.

(i) Parametrize the smooth surface that is the paraboloid \( z = x^2 + y^2 \) where \( z \leq 9 \).

\[
x = r \cos \theta, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 3
\]

\[
y = r \sin \theta
\]

\[
z = r^2
\]

\[
0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 3
\]
2. (a) (5 pts) Let \( w = xy + \ln z \) where \( x = \frac{u^2}{v} \), \( y = u + v \) and \( z = \cos u \). Find \( \frac{\partial w}{\partial v} \) when \( u = -1 \) and \( v = 2 \).

\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v} \\
= \left( y \cdot \frac{2v}{u} \right) + (u + v) \cdot 1 \\
= \left( -1 \right) \cdot \frac{4}{-1} + \frac{4}{-1} \cdot 1 = -8
\]

(b) (5 pts) Calculate the directional derivative of the function \( f(x, y) = 2xy - y^3 \) in the direction \( (3, -4) \) at the point \((1, 3)\)

\[
\vec{u} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad \nabla f(1, 3) = \begin{bmatrix} 2y \\ 2x - 3y^2 \end{bmatrix} \bigg|_{(1, 3)} = \begin{bmatrix} 6 \\ -25 \end{bmatrix} \\
D_u f(1, 3) = \vec{u} \cdot \nabla f(1, 3) \\
= \begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -25 \end{bmatrix} = \frac{18 - 100}{5} = \frac{-82}{5}
\]

3. (a) (5 pts) Write an equation for a plane through the point \((x_0, y_0, z_0)\) with normal vector \( \vec{n} = (a, b, c) \). Then with the help of a picture, use the dot product to explain why your equation is correct.

\[
\left\{ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \right\}
\]

\( \vec{n} = \langle a, b, c \rangle \) and \( \vec{V} \) are perpendicular for any \((x, y, z)\) in plane

\[
\vec{n} \cdot \vec{V} = 0 \\
\Rightarrow \quad \langle a, b, c \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0
\]

(b) (5 pts) Find an equation of a tangent plane to the surface \( z^2 + y^2 - z^2 = 0 \) at the point \((-3, 4, -5)\).

\[
\vec{n} = \nabla f(-3, 4, -5) = \left\{ \frac{2x, 2y, -2z}{-2, 4, -10} \right\} = \langle -6, 8, 10 \rangle \\
\Rightarrow \quad \vec{n} = \langle 3, -4, -5 \rangle \quad \text{is a normal vector.}
\]

\[
\text{plane: } 2x - 4y + 5z = 0
\]
4. (10 pts) Find the point where the function \( f(x, y, z) = 3x^2 + y^2 + 2z^2 \) attains its minimum when restricted to the plane \( 2x + y - z = 17 \).

Using Lagrange multipliers:

\[
\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 
6x = 2\lambda \\
2y = \lambda \\
4z = -\lambda 
\end{cases}
\Rightarrow \begin{cases} 
\lambda = 3x \\
\lambda = 2y \\
\lambda = -4z 
\end{cases}
\Rightarrow \begin{cases} 
x = \frac{\lambda}{3} \\
y = \frac{\lambda}{2} \\
z = \frac{\lambda}{4} 
\end{cases}
\]

\[
\Rightarrow \frac{4}{3}y + y - \left(-\frac{1}{2}z\right) = 17 \Rightarrow \left(\frac{8}{6} + \frac{6}{6} + \frac{3}{6}\right)y = 17
\Rightarrow \frac{17}{6}y = 17 \Rightarrow y = 6, \quad x = 4, \quad z = -3
\]

\( (4, 6, -3) \)

5. (10 pts) Evaluate the integral \( \int_0^1 \int_{y/2}^1 e^{-x^2} \, dx \, dy \) by reversing the order of integration.

\[
\int_0^1 \int_{y/2}^1 e^{-x^2} \, dx \, dy
= \int_0^{1} \int_0^{2x} e^{-x^2} \, dy \, dx
= \int_0^1 2x \, e^{-x^2} \, dx
= \left[ -e^{-x^2} \right]_0^1 = 1 - e^{-1}
\]
6. (10 pts) Consider the planar lamina bounded by \( x = 16 - y^2 \) and \( x = -9 \) whose density is given by \( \rho = kx^2y^2 \).

(a) Write the formulas for \( \bar{x} \) and \( \bar{y} \) in terms of \( M, M_x \) and \( M_y \).

(b) Write the integrals needed to find \( M, M_x \) and \( M_y \) for this lamina, but do not evaluate the integrals.

\[
\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}
\]

\[
M = \int_{-5}^{5} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} kx^2y^2 \, dx \, dy
\]

\[
M_x = \int_{-5}^{5} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} kx^3y^2 \, dx \, dy
\]

\[
M_y = \int_{-5}^{5} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} kx^2y^3 \, dx \, dy
\]

7. (10 pts) Sketch the solid whose volume is given by \( \int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{\sqrt{9-x^2}-y^2} \, dz \, dy \, dx \) and evaluate the integral by converting it to either cylindrical or spherical coordinates (whichever looks easier).

\[
V = \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{r} r^2 \, dz \, dr \, d\theta
\]

\[
= \int_{0}^{\pi} \int_{0}^{\pi} \left[ (4-r^2) \right] \, r \, dr \, d\theta
\]

\[
= \int_{0}^{\pi} \left[ \frac{q}{2} - \frac{q^4}{4} \right] \, d\theta
\]

\[
= \pi \cdot \frac{q}{4} = \frac{\pi q}{4}
\]
8. (a) (5 pts) Find the surface area of a wall whose height is \( h(x, y) = x \) along the circle \( x^2 + y^2 = 1 \) where \( x \geq 0 \).

\( b) \) (5 pts) Evaluate the line integral \( \int_C x^2 \, dy - y \, dx \) where \( C \) is the portion of \( x = y^2 \) that starts at \((4, -2)\) and ends at \((1, 1)\).

\[
\begin{align*}
\text{b)} \quad &x = t^2, \quad y = t, \quad t = -2 \ldots 1 \\
\int_C x^2 \, dy - y \, dx &= \int_{t=-2}^{t=1} t^4 \, dt - t(2t \, dt) \\
&= \int_{t=-2}^{t=1} \left( t^4 - 2t^2 \right) \, dt \\
&= \left[ \frac{t^5}{5} - \frac{2t^3}{3} \right]_{t=-2}^{t=1} \\
&= \left( \frac{1}{5} - \frac{2}{3} \right) - \left( \frac{-32}{5} + \frac{16}{3} \right) \\
&= \frac{33}{5} - \frac{18}{3} = \frac{3}{5}
\end{align*}
\]

9. Let \( S \) be the portion of the plane \( 2x + 3y + 2z = 12 \) that is inside the cylinder \( x^2 + y^2 = 16 \).

(a) (5 pts) Set-up but do not evaluate a surface integral to find the mass of the surface \( S \) given that its density per unit area is \( \rho(x, y, z) = x^2 + y^2 + z^2 \).

(b) (5 pts) Set up but do not evaluate a surface integral to find the flux of the vector field \( \mathbf{F} = 2xi - xyj + 5k \) through the surface \( S \) oriented with upward unit normal.

\[
\begin{align*}
\text{a)} \quad &\mathbf{m} = \int_D \int (x^2 + y^2 + z^2) \, dS \\
&= \int_{0}^{2\pi} \int_{0}^{4} \left[ r^2 + \left( b - \frac{3}{2} \right) r \cos \theta \right] \sqrt{1 + \left( \frac{3}{2} \right)^2 + 1} \, r \, dr \, d\theta \\
&= \int_{0}^{2\pi} \int_{0}^{4} \left[ r^2 + \left( 6 - \frac{3}{2} \right) r \cos \theta \right] \sqrt{r^2 + 6 - \frac{3}{2} \cos \theta} \, \frac{r^2}{2} \\
&= \int_{0}^{2\pi} \int_{0}^{4} \left[ 2r \cos \theta - \frac{3r^2}{2} \cos \theta + 5 \right] \, r \, dr \, d\theta \\
\end{align*}
\]

\[
\begin{align*}
\text{b)} \quad &\mathbf{N} \cdot d\mathbf{S} = \left< 1, \frac{3}{2}, 1 \right> \, dA \\
&\text{since} \quad \mathbf{G}(x, y, z) = z - (6 - \frac{3}{2} y - x) = 0 \\
&\text{Flux} = \int_D \int \left< 2x, -xy, 5 \right> \left< 1, \frac{3}{2}, 1 \right> \, dA = \int_D \int \left( 2x - \frac{3}{2} xy + 5 \right) \, dA \\
&= \int_{0}^{2\pi} \int_{0}^{4} \left( 2r \cos \theta - \frac{3r^2}{2} \cos \theta + 5 \right) \, r \, dr \, d\theta
\end{align*}
\]
10. (10 pts) Let $S$ be the surface of the rectangular box $-1 \leq x \leq 2, 0 \leq y \leq 3$ and $0 \leq z \leq 5$ oriented with outward unit normal. Let $F(x, y, z) = 4xi + yj + (2z - 1)k$.

(a) Find the flux of $F$ through the closed surface $S$.

(b) Use Stokes' theorem evaluate line integral $\int_C F \cdot dr$ as a double integral, where $F$ is as above, and $C$ is the boundary of the top side of the box described above oriented in a counter clockwise direction when viewed from above.

\[ \text{a) By the divergence theorem, } \text{div } F = 4 + 0 + 2 = 6 \]

\[ \text{Flux} = \iiint_S \text{div } F \, dV = \iiint_Q 6 \, dV = 6 \cdot \text{Volume } Q = 9 \cdot 30 = 270 \]

\[ \text{b) } \int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{Curl } \vec{F}) \cdot \hat{N} \, dS \]

\[ \text{Curl } \vec{F} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
3 & 2 & 1 \\
4 & 0 & 2z - 1
\end{vmatrix} = -\hat{O}_x + 0\hat{y} + 1\hat{z} \]

\[ \hat{N} \, dS = \hat{k} \, dA \]

\[ \text{Area of } D = 3 \cdot 3 = 9 \]

**Extra Credit.** On the back of this sheet, derive the equation for the "least squares" line of best fit to the data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.