Math 233, Test 3, Spring 2006

Name: 

Instructions. Do the first question, and any four of the last five questions. Please do your best, and show all appropriate work. Thank you.

1. (18 pts) (a) Write the formula for \( \text{div}\vec{F} \) where \( \vec{F}(x,y,z) = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k \)

\[
\text{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}
\]

(b) Write the formula for \( \text{curl}\vec{F} \) where \( \vec{F} \) is as in (a). If you use determinant notation, expand the determinant.

\[
\text{curl}\vec{F} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
M & N & P
\end{vmatrix} = (\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})\hat{i} + (\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x})\hat{j} + (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\hat{k}
\]

(c) Parametrize the line segment that starts at \((0, 5)\) and ends at \((2, -1)\).

\[
\text{Infinitely many correct answers:} \quad \begin{align*}
x &= 2t, & 0 \leq t \leq 1 \\
y &= 5 - 6t
\end{align*}
\]

(d) Given \( f(x,y) = x^2y \), evaluate the line integral \( \int_C \nabla f \cdot d\vec{r} \) where \( C \) is the parabola \( x = y^2 \) starting at \((1, -1)\) and ending at \((4, 2)\).

\[
\int_C \nabla f \cdot d\vec{r} = \int_{(1,-1)}^{(4,2)} f(x,y) \left| \begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
1 & 0
\end{array} \right| = x^2y \left| \begin{array}{cc} 1 & 0 \\
1 & 0
\end{array} \right| = 32 - (-1) = 33
\]

(e) Suppose \( C \) is a closed piecewise smooth simply connected curve that encloses the region \( R \) where \( C \) is oriented in a counterclockwise direction. Show that \( \int_C -y\,dx \) is the area of \( R \).

\[
\int_C -y\,dx = \iint_R \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) dA = \iint_R -(-1)\,dA = \iint_R dA = \text{Area of } R.
\]

(f) Using the fundamental theorem of line integrals and potential functions, explain why a conservative vector field is path independent.

By FTLI, the value of the line integral depends only on the values of the potential function at the endpoints, not on the path between the endpoints.
2. (8 pts) (a) Sketch the vector field \( \mathbf{F}(x, y) = -\frac{y}{x} \mathbf{i} + y \mathbf{j} \) by sketching some representative vectors.

(b) Is \( \mathbf{F} \) conservative?

(c) Find a potential function for \( \mathbf{F} \) or explain why no potential function exists.

\[ b) \quad \text{Yes, } \frac{\partial y}{\partial x} = 0 = \frac{\partial (-y/x)}{\partial y} \] 

[or by (c), it has a potential function, i.e., \( \mathbf{F} \) is conservative.]

\[ c) \quad f(x, y) = -\frac{x^2}{4} + \frac{y^2}{2} + k \]

is a potential function for any constant \( k \).

3. (8 pts) Use a line integral to find the surface area of a retaining wall whose height is given by \( 2y \), and the base of the wall is on the top half of the circle \( x^2 + y^2 = 64 \).

\[ C: \quad x(t) = 8 \cos t \quad \text{for} \quad 0 \leq t \leq \pi \]

\[ y(t) = 8 \sin t \]

\[ ds = \sqrt{\left(8 \cos t\right)^2 + \left(8 \sin t\right)^2} \, dt = \sqrt{64 \left(\sin^2 t + \cos^2 t\right)} \, dt = 8 \, dt \]

\[ \text{Area} = \int_C 2y \, ds \]

\[ = \int_0^\pi 2(8 \sin t) \, 8 \, dt \]

\[ = 128 \int_0^\pi \sin t \, dt \]

\[ = 128 \left[ -\cos t \right]_0^\pi \]

\[ = 128 \left[ -(1) - (-1) \right] = 256 \]
4. (8 pts) Let \( C \) be the triangle with vertices (0,0), (0,5) and (5,10) oriented in a counterclockwise direction. Set-up the integral(s) to calculate the work done by the force field \( \mathbf{F}(x,y) = y^2i + xj \) along \( C \). Do not evaluate the integrals, but express the work as a sum of integrals that a Calculus II student could evaluate.

\[
W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C y^2 \, dx + x \, dy
\]

\[
= \int_{C_1} (4t^2 \, dt + t(2t \, dt) + \int_{C_2} (t+5)^2 \, dt + t \, dt + \int_{C_3} t^2 \, dt + 0 \, dt
\]

\[
= \int_0^5 (4t^2 + 2t) \, dt - \int_0^5 (t^2 + 11t + 25) \, dt
\]

Check with #5: \( \int_0^5 (t^2 - 9t - 25) \, dt = t^3 - \frac{9t^2}{2} - 25t \bigg|_0^5 = 125 - \frac{225}{2} - 125 - 0 = -\frac{225}{2} \)

5. (8 pts) Use Green's theorem to calculate the work done by the vector field along the curve \( C \) in question 4.

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial}{\partial x} [x] - \frac{\partial}{\partial y} [y^2] \right) \, dA = \int_0^5 \int_0^{x+5} (1-2y) \, dy \, dx
\]

\[
= \int_0^5 \left( y - y^2 \right) \bigg|_{2x}^{x+5} \, dx = \int_0^5 (x+5) - (x+5)^2 - (2x-x^2) \, dx
\]

\[
= \int_0^5 (x+5 - x^2 - 10x - 25 - 2x + 4x^2) \, dx = \int_0^5 (3x^2 - 11x - 20) \, dx = x^3 - \frac{11x^2}{2} - 20x \bigg|_0^5
\]

\[
= 125 - \frac{225}{2} - 100 - 0 = 25 - \frac{225}{2} = -\frac{225}{2}
\]
6. (8 pts) (a) Find a potential function for the vector field \( \mathbf{F}(x, y, z) = (y + 2z)i + (x - 3z)j + (2x - 3y)k \).

(b) Evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} \) is the vector field in (a), and \( C \) is the curve formed by the line segments from \((0,0,0)\) to \((1,0,1)\) to \((-1,1,1)\) to \((1,2,3)\).

\[ a) \text{ Potential function: } f(x, y, z) = xy + 2xz - 3yz + k \]

\[ b) \text{ By FTLT} \]
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \left. xy + 2xz - 3yz \right|_{(0,0,0)}^{(1,2,3)} \]
\[ = (1)(2) + 2(1)(3) - 3(2)(3) \]
\[ = -10 \]

**Bonus.** (+4pts) Prove the fundamental theorem of line integrals.

*See Text §14.3, p. 1033.*