Math 233, Test 1, Spring 2007

Instructions. Do the first question, and any six of the last seven questions. Please do your best, and show all appropriate work. Thank you.

1. (14 pts) (a) Write a formula for $\frac{\partial F}{\partial v}$ for the differentiable function $F(x, y, z)$ where $x, y,$ and $z$ are differentiable functions of $u$ and $v$.

(b) Find the domain of $f(x, y) = \ln(9 - x^2 - y^2)$.

(c) State the definition for the directional derivative of $f$ at a point $(x, y)$ in the direction of a unit vector $u = \langle a, b \rangle$.

(d) Is $\nabla f(x, y)$ normal to the surface $z = f(x, y)$? If not, what might it be normal to?

(e) What are the conditions that define a critical point for a function $f(x, y)$?

(f) What conditions on the partial derivatives of $f$ guarantee that $f$ is differentiable? Is it enough for the partial derivatives to exist?

(g) What is the maximum rate of increase of the function $f(x, y) = 3x^2y - y$ at the point $(2, 3)$?
2. (5 pts) Does the limit \( \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} \) exist? Hint: try the paths \( x = 0 \) and \( x = y^2 \) and explain your conclusion.

3. (5 pts) Suppose the equation \( x(x^2 + y^2) - y^2 = 6 \) implicitly defines \( y \) as a differentiable function of \( x \). Find \( \frac{dy}{dx} \).

4. (5 pts) Calculate the directional derivative of the function \( f(x, y) = x^3 - 3xy \) in the direction \((-5, 12)\) at the point \((1, 3)\)
5. (5 pts) A cargo container (in the shape of a rectangular solid) must have a volume of 600 cubic feet. The bottom needs to be stronger, so it will cost $6 per square foot to construct, whereas the sides and top will cost $3 per square foot to construct. Find the dimensions of the container of this size that has minimum cost.

6. (5 pts) Find the maximum and minimum values of the function \( f(x, y) = 3x^2 + 2y^2 - 4y \) over the region in the \( xy \)-plane bounded by the graphs of \( y = x^2 \) and \( y = 9 \).
7. (5 pts) Find and classify the critical points of \( f(x, y) = 4xy - x^4 + y^4 \).

8. (5 pts) Find the equation of the tangent plane to the surface \( z = x^2 - y^2 \) at the point where \( x = 3 \) and \( y = -1 \).

Extra Credit. On the back of this sheet, derive the equation for the “least squares” line of best fit to the data \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\).