Math 233: Test II, Spring 2007 – Multiple Integrals

Name: ________________________________

Instructions. Do question **1** and any **4** of questions **2** through **6**. Please justify all answers and **do not use a calculator**.

1. (15 pts) (a) Sketch the region of integration for $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} y\,dy\,dx$.

Answer. The region is top half of circle of radius 3 centered at the origin—the sketch is left to you.

(b) Use geometry to determine value of integral $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} 1\,dy\,dx$. (Same limits as (a))

Answer. The integral is the area of the region in (a) which is $\frac{9\pi}{2}$.

(c) Write the double integral from (a) using polar coordinates.

Answer. $\int_{0}^{\pi}\int_{0}^{3} r^2\sin\theta\,dr\,d\theta$

(d) The integral given in cylindrical coordinates as $\int_{0}^{2\pi}\int_{0}^{2}\int_{0}^{10} r dz\,dr\,d\theta$ represents a volume of a solid. Describe the solid.

Answer. This represents the volume of a cylinder of radius 2 centered on the z-axis with a height of 10.

(e) Set-up but do not evaluate an integral in spherical coordinates find the volume of the solid ball of radius 4 centered at the origin.

Answer. $V = \int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{4} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$

2. (10 pts) Evaluate the integral $\int_0^4 \int_{x/2}^2 e^{y^2}\, dy\, dx$ by reversing the order of integration.

Answer.

$$\begin{aligned}\int_0^4 \int_{x/2}^2 e^{y^2}\, dy\, dx &= \int_0^2 \int_0^{2y} e^{y^2}\, dx\, dy \\ &= \int_0^2 2ye^{y^2}\, dy = e^{y^2}\Big|_0^2 = e^4 - 1.\end{aligned}$$

3. (10 pts) Consider the planar lamina bounded by $x = 2 - y^2$ and $y = x$ whose density is given by $\rho = ky^2$.

(a) Write the formulas for $\bar{x}$ and $\bar{y}$ in terms of M, M_x and M_y.

Answer. $\bar{x} = \dfrac{M_y}{M}, \quad \bar{y} = \dfrac{M_x}{M}.$

(b) Write the integrals needed to find M, M_x and M_y for this lamina, but do not evaluate the integrals.

Answer. The region is horizontally simple (sketch picture):

$$M = \int_{-2}^1 \int_y^{2-y^2} ky^2\, dx\, dy$$

$$M_x = \int_{-2}^1 \int_y^{2-y^2} ky^3\, dx\, dy \qquad \text{and} \qquad M_y = \int_{-2}^1 \int_y^{2-y^2} kxy^2\, dx\, dy$$

4. (10 pts) Use integration to find the surface area of the portion of the cone $z = 3\sqrt{x^2 + y^2}$ that lies between the planes $z = 3$ and $z = 12$.

Answer. First $f(x, y) = 3\sqrt{x^2 + y^2}$ and so $f_x = \dfrac{3x}{\sqrt{x^2 + y^2}}$ and $f_y = \dfrac{3y}{\sqrt{x^2 + y^2}}$. Then $\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{10}$. Now, surface area is given by

$$\int\int_R \sqrt{f_x^2 + f_y^2 + 1}\, dA = \int\int_R \sqrt{10}\, dA = \sqrt{10} \cdot \text{area of } R$$

where R is the region in the xy-plane that is within the circle $x^2 + y^2 = 16$ and outside the circle $x^2 + y^2 = 1$. Thus R has area $\pi(4^2 - 1^2) = 15\pi$, and thus the area of the desired surface is $15\sqrt{10}\pi$.

5. (10 pts) Evaluate $\displaystyle\int\int_R (x - y)(x + 4y)^2\, dA$ where R is the parallelogram with vertices $(0, 0)$, $(2, 2)$, $(6, 1)$ and $(4, -1)$.

Hint: You may wish to use the change of variables $u = x - y$ and $v = x + 4y$.

Answer. Notice that R is bounded by the lines $x + 4y = 0$, $x + 4y = 10$, $x - y = 0$ and $x - y = 5$. This corresponds in the uv-plane to the region $0 \le u \le 5$ and $0 \le v \le 10$. Solving $u = x - y$ and $v = x + 4y$ for x and y leads to

$$x = \frac{4}{5}u + \frac{1}{5}v \qquad \text{and} \qquad y = -\frac{1}{5}u + \frac{1}{5}v.$$

The Jacobian is thus

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{4}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{vmatrix} = \frac{4}{25} + \frac{1}{25} = \frac{1}{5}.$$

Consequently,

$$\begin{aligned}
\int\int_R (x - y)(x + 4y)^2\, dA &= \int_0^5 \int_0^{10} uv^2 \left|\frac{1}{5}\right| dv\, du \\
&= \frac{1}{5} \int_0^5 u \frac{v^3}{3}\Big|_0^{10} du \\
&= \frac{1000}{15} \int_0^5 u\, du = \frac{1000}{15} \cdot \frac{u^2}{2}\Big|_0^5 \\
&= \frac{5000}{6} = \frac{2500}{3}.
\end{aligned}$$

6. (a) (5 pts) Evaluate the integral $\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{0}^{5} x\,dz\,dx\,dy$; use any method you wish.

Answer. You many integrate this directly, it is not hard. It is slightly shorter in cylidrical coordinates:

$$\begin{aligned}\int_0^2\int_{-\pi/2}^{\pi/2}\int_0^5 r^2\cos\theta\,dz\,d\theta\,dr &= \int_0^2\int_{-\pi/2}^{\pi/2} 5r^2\cos\theta\,d\theta\,dr \\ &= \int_0^2 10r^2\,dr = \frac{80}{3}.\end{aligned}$$

(b) (5 pts) Rewrite the integral $\int_0^4\int_z^4\int_0^{\frac{1}{2}\sqrt{y^2-z^2}} xyz\,dx\,dy\,dz$ using the order $dz\,dy\,dx$. Do not evaluate the integral.

Answer. Sketch the region, and notice $x = \frac{1}{2}\sqrt{y^2-z^2}$ implies $z = \sqrt{y^2-4x^2}$. The integral in the requested order is thus

$$\int_0^2\int_{2x}^4\int_0^{\sqrt{y^2-4x^2}} xyz\,dz\,dy\,dx$$