Name: Hints and Answers

Instructions. Do the first question, and any six of the last seven questions. Please do your best, and show all appropriate work. Thank you.

1. (14 pts) (a) Write a formula for \( \frac{\partial F}{\partial w} \) for the differentiable function \( F(x, y) \) where \( x \) and \( y \) are differentiable functions of \( u, v \) and \( w \).

Answer. \( \frac{\partial F}{\partial w} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial w} \).

(b) Find the domain of \( f(x, y) = \frac{1}{\sqrt{x^2 + y^2 - 4}} \).

Answer. \( \{(x, y) : x^2 + y^2 > 4\} \).

(c) State the definition for the directional derivative of \( f \) at a point \( (x, y) \) in the direction of a unit vector \( u = \langle a, b \rangle \).

Answer. \( D_u f(x, y) = \lim_{t \to 0} \frac{f(x + ta, y + tb) - f(x, y)}{t} \) provided the limit exists.

(d) Find a vector that is normal to the surface \( z^2 - y^2 - x^2 = 4 \) at the point \( (1, 2, -3) \).

Answer. \( \nabla F(x, y, z) = \langle -2x, -2y, 2x \rangle \); therefore a normal vector is \( \nabla F(1, 2, -3) = \langle -2, -4, -6 \rangle \) or any nonzero multiple thereof.

(e) In terms of limits, when is a function \( f(x, y) \) continuous at a point \( (a, b) \)?

Answer. In short, if \( f(a, b) = \lim_{(x,y) \to (a,b)} f(x, y) \); this means: (i) \( (a, b) \) is in the domain of \( f \); (ii) the limit \( \lim_{(x,y) \to (a,b)} f(x, y) \) exists; and (iii) that limit is equal to \( f(a, b) \).

(f) In which unit vector direction is the function \( f(x, y) = x^2 + 3xy \) increasing the fastest at the point \( (1, -2) \)?

Answer. \( \frac{\nabla f(1, -2)}{\|\nabla f(1, -2)\|} = \frac{\langle -4, 3 \rangle}{\|\langle -4, 3 \rangle\|} = \frac{-4}{5} \mathbf{i} + \frac{3}{5} \mathbf{j} \).

(g) Suppose \( \lim_{(x,y) \to (a,b)} f(x, y) = 8 \) and \( \lim_{(x,y) \to (a,b)} g(x, y) = -2 \). Find \( \lim_{(x,y) \to (a,b)} (fg)(x, y) \) and \( \lim_{(x,y) \to (a,b)} (f - g)(x, y) \).

Answer. \( \lim_{(x,y) \to (a,b)} (fg)(x, y) = (8)(-2) = -16; \lim_{(x,y) \to (a,b)} (f - g)(x, y) = 8 - (-2) = 10. \)
2. (5 pts) Does the limit \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} \) exist? Hint: try the limit along various lines \( y = mx \) and explain your conclusion.

\textbf{Answer.} Along a path \( y = mx \) we compute

\[
\lim_{x \to 0} \frac{x(mx)}{x^2 + (mx)^2} = \lim_{x \to 0} \frac{m}{1 + m^2}.
\]

Thus the limit depends on the path chosen. For example, if \( m = 0 \), the limit along that path is 0, and if \( m = 1 \), the limit along that path is 1/2. Consequently, the limit does not exist because different paths result in different limits.

3. (5 pts) Suppose the equation \( y(x^2 + y^2) - x^2 = 9 \) implicitly defines \( y \) as a differentiable function of \( x \). Find \( \frac{dy}{dx} \) when \( x = 1 \) and \( y = 2 \).

\textbf{Answer.} Let \( F(x, y) = x^2y + y^3 - x^2 \). Then

\[
\frac{dy}{dx} = -F_x = -\frac{2xy - 2x}{x^2 + 3y^2}.
\]

Therefore, \( \frac{dy}{dx} = -\frac{2}{13} \) when \( x = 1 \) and \( y = 2 \).

4. (5 pts) Suppose \( f \) is a differentiable function of \( x \) and \( y \). Using the formula \( D_u f(x, y) = \nabla f(x, y) \cdot u \) and properties of dot products, explain why \( f \) decreases the fastest at \( (x, y) \) in the direction \( -\nabla f(x, y) \), and why the rate of decrease is \( -\|\nabla f(x, y)\| \).

\textbf{Answer.} Using properties of dot products,

\[
D_u f(x, y) = \cos \theta \|\nabla f(x, y)\| \|u\| = \cos \theta \|\nabla f(x, y)\|
\]

because \( \|u\| = 1 \) and where \( \theta \) is the angle between \( u \) and \( \nabla f(x, y) \). The smallest possible value occurs when \( \cos \theta = -1 \). This means \( \theta = 180^\circ \), so \( u \) is in the opposite direction of \( \nabla f(x, y) \), that is in the same direction as \( -\nabla f(x, y) \) and the value of the directional derivative in this direction is \( -\|\nabla f(x, y)\| \).
5. (5 pts) Find the point on the plane \(3x - 2y + z = 28\) that is closest to \((0,0,0)\).

**Answer.** Minimize \(f(x, y, z) = x^2 + y^2 + z^2\) subject to \(g(x, y, z) = 3x - 2y + z = 28\). Using Lagrange multipliers one solves \(\nabla f = \lambda \nabla g\), or

\[
2x = 3\lambda \quad 2y = -2\lambda \quad 2z = \lambda
\]

The first and third equation imply \(x = 3z\); the second and third imply \(y = -2z\). When these relations are substituted back into the constraint we obtain

\[9z + 4z + z = 28.\]

This implies \(z = 2\). The nearest point is thus \((6, -4, 2)\).

6. (5 pts) The centripetal acceleration of a particle moving in a circle is given by \(a = \frac{v^2}{r}\) where \(v\) is the velocity and \(r\) is the radius of the circle. Use the differential of \(a\) to approximate the maximum percent error in measuring the acceleration due to errors of 3% in \(v\) and 2% in \(r\).

**Answer.** We compute

\[
da = a_v dv + a_r dr = \frac{2v}{r} dv - \frac{v^2}{r^2} dr
\]

\[= \frac{2v}{r} (\pm .03v) - \frac{v^2}{r^2} (\pm .02r).\]

The maximum value of \(da\) is then

\[da = .06 \frac{v^2}{r} + .02 \frac{v^2}{r} = .08 \frac{v^2}{r} = .08a.\]

Thus there is an estimated maximum possible percentage error in measuring of 8%. 
7. (5 pts) Find and classify the relative extrema of \( f(x, y) = x^5 - 5xy + \frac{5}{2}y^2 \).

**Answer.** First, we find the critical points \( f_x = 5x^4 - 5y = 0 \) and \( f_y = -5x + 5y = 0 \). The second equation implies \( y = x \); substituting this into the first implies \( 5x^4 - 5x = 0 \), and so \( x(x^3 - 1) = 0 \) which implies \( x = 0 \) or \( x = 1 \). Thus the critical points are \((0, 0)\) and \((1, 1)\).

Next, \( f_x = 20x^3 \), \( f_{yy} = 5 \) and \( f_{xy} = -5 \). Therefore,
\[
D = f_{xx}f_{yy} - f_{xy}^2 = 80x^3 - 25.
\]

At the critical point \((0, 0)\), \( D = -25 < 0 \) and so \( f \) has a saddle at \((0, 0)\).

At the critical point \((1, 1)\), \( D = 55 > 0 \), and \( f_{xx} = 20 > 0 \), and so \( f \) has a relative minimum at \((1, 1)\).

8. (5 pts) Find equations for the normal line to the surface \( z = x^3 - 3y^2 \) at the point where \( x = 1 \) and \( y = 2 \).

**Answer.** Let \( F(x, y, z) = x^3 - 3y^2 - z = 0 \). Then \( \nabla F(x, y, z) = \langle 3x^2, -6y, -1 \rangle \). When \( x = 1 \) and \( y = 2 \), we find \( z = 1 - 12 = -11 \). Therefore, the direction of the normal line is \( \nabla F(1, 2, -11) = \langle 3, -12, -1 \rangle \) and it has a parametric representation
\[
x = 1 + 3t, \quad y = 2 - 12t \quad z = -11 - t.
\]

If you prefer, a representation using symmetric equations is
\[
\frac{x - 1}{3} = \frac{y - 2}{-12} = \frac{z + 11}{-1}.
\]

**Extra Credit.** On the back of this sheet, derive the equation for the “least squares” line of best fit to the data \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\).