Name: ____________________________

Hints and Answers

Instructions. Do question 1 and any four of questions 2 through 6. Please justify all answers and do not use a calculator.

1. (10 pts) (a) Sketch the region of integration for \( \int_{-2}^{0} \int_{-y}^{y+4} f(x, y) \, dx \, dy \).

Answer. The region is a triangle with vertices \((0, 0)\), \((2, -2)\) and \((4, 0)\).

(b) Use geometry to determine value of integral \( \int_{-2}^{0} \int_{-y}^{y+4} 1 \, dx \, dy \). (Same limits as (a))

Answer. The triangle in (a) has base 4 and height 2, thus its area, which is the value of the integral requested, is 4.

(c) The triple integral in spherical coordinates \( \int_{0}^{2\pi} \int_{0}^{\pi} \int_{2}^{5} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \) represents the volume of a solid. Describe the solid.

Answer. The solid that is outside a sphere of radius 2 centered at the origin and inside the sphere of radius 5 centered at the origin.

(d) Set-up but do not evaluate an integral in cylindrical coordinates to find the volume of a right circular cylinder of height 5 and radius 2.

Answer. \( V = \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{5} r \, dz \, dr \, d\theta \).

(e) Let \( x = 2u - v \) and \( y = 3u + v \). Find \( \frac{\partial(x, y)}{\partial(u, v)} \).

Answer. \( \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (2)(1) - (3)(-1) = 5 \).
2. (10 pts) Evaluate the integral \( \int_0^1 \int_y^1 \sin(x^2) \, dx \, dy \) by reversing the order of integration.

**Answer.** Sketch the region as needed to help make the switch:

\[
\int_0^1 \int_y^1 \sin(x^2) \, dx \, dy = \int_0^1 \int_0^x \sin(x^2) \, dy \, dx
\]

\[
= \int_0^1 x \sin(x^2) \, dx \quad u = x^2, \quad du = 2x \, dx
\]

\[
= \frac{1}{2} \int_0^1 \sin u \, du = -\frac{1}{2} \cos u \bigg|_0^1
\]

\[
= \frac{1 - \cos(1)}{2}.
\]

3. (10 pts) Consider the planar lamina bounded by \( x = 16 - y^2 \) and \( x = 0 \) whose density is given by \( \rho = kx \). (a) Sketch the lamina, then set-up but do not evaluate the integrals to find: (b) the mass \( M \), (c) the moment \( M_x \), and (d) the moment \( M_y \).

(a) The sketch is left to the reader.

(b) \( M = \int_{-4}^4 \int_0^{16-y^2} kx \, dx \, dy. \)

(c) \( M_x = \int_{-4}^4 \int_0^{16-y^2} kxy \, dx \, dy. \)

(d) \( M_y = \int_{-4}^4 \int_0^{16-y^2} kx^2 \, dx \, dy. \)
4. (10 pts) Find the surface area of the portion of the graph \( z = 9 + x^2 - y^2 \) that lies above the region \( R = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\} \).

**Answer.** The surface area formula is

\[
\int \int_R \sqrt{1 + (f_x)^2 + (f_y)^2} \, dA.
\]

Therefore,

\[
\text{Surface area} = \int \int_R \sqrt{1 + (2x)^2 + (-2y)^2} \, dA
\]

\[
= \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta \quad u = 1 + 4r^2, \ du = 8r \, dr
\]

\[
= \int_0^{2\pi} \left[ \frac{1}{3} \cdot \frac{2}{3} u^{3/2} \right]_1^5 \, d\theta
\]

\[
= 2\pi \cdot \frac{1}{8} \cdot \frac{2}{3} [17^{3/2} - 5^{3/2}] = \frac{\pi}{6} [17^{3/2} - 5^{3/2}].
\]

5. (10 pts) Evaluate \( \int \int_R (x - y)\sqrt{(x + y)} \, dA \) where \( R \) is the parallelogram with vertices \((0, 0), (-1, 1), (2, 4)\) and \((3, 3)\). by using the change of variables \( u = x - y \) and \( v = x + y \).

**Answer.** The region \( R \) is a parallelogram with sides \( x + y = 0, x + y = 6, x - y = -2 \) and \( x - y = 0 \). Thus with the transformation \( u = x - y \) and \( v = x + y \) the region \( S \) in the \( uv \)-plane is \( S = \{(u, v) : -2 \leq u \leq 0, 0 \leq v \leq 6 \} \). Solving \( u = x - y \) and \( v = x + y \) for \( x \) and \( y \) (add and subtract equations respectively) results in

\[
x = \frac{1}{2} u + \frac{1}{2} v \quad \text{and} \quad y = -\frac{1}{2} u + \frac{1}{2} v.
\]

The Jacobian is then

\[
\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = 1.
\]

Using the change of variable formula one obtains

\[
\int \int_R (x - y)\sqrt{(x + y)} \, dA = \int_0^6 \int_{-2}^0 u^{1/2} \frac{1}{2} \, du \, dv = \int_0^6 \frac{u^2}{4} v^{1/2} \bigg|_{u=-2}^{u=0} \, dv = \int_0^6 -v^{1/2} \, dv
\]

\[
= -\frac{2}{3} v^{3/2} \bigg|_0^6 = -\frac{2}{3} \cdot 6^{3/2} = -4\sqrt{6}.
\]
6. (a) (5 pts) Evaluate the integral \( \int_{-2}^{0} \int_{0}^{\sqrt{4-z^2}} \int_{0}^{5} x \, dz \, dy \, dx \); use any method you wish.

**Answer.** My preferred method is cylindrical coordinates, or to evaluate directly for the first step, and then switch to polar. Notice that the region of integration projected on the xy-plane is the portion enclosed by a circle of radius 2 centered at the origin that lies in the 2nd quadrant. Hence the limits on \( \theta \) are \( \pi/2 \leq \theta \leq \pi \) and the limits for \( r \) are \( 0 \leq r \leq 2 \).

\[
\int_{-2}^{0} \int_{0}^{\sqrt{4-z^2}} \int_{0}^{5} x \, dz \, dy \, dx = \int_{\pi/2}^{\pi} \int_{0}^{2} \int_{0}^{5} r \cos \theta \, r \, dr \, d\theta \, dz = \int_{\pi/2}^{\pi} \int_{0}^{2} r^2 \cos \theta \, r \, dr \, d\theta = \int_{\pi/2}^{\pi} 5r^3 \cos \theta \, d\theta = \int_{\pi/2}^{\pi} \frac{40}{3} \cos \theta \, d\theta = \frac{40}{3} \sin \theta \bigg|_{\pi/2}^{\pi} = \frac{40}{3} (0 - 1) = -\frac{40}{3}.
\]

(b) (5 pts) Rewrite the integral \( \int_{0}^{1} \int_{0}^{1-y^2} \int_{0}^{1-y} xyz \, dz \, dx \, dy \) using the order \( dx \, dy \, dz \). Do not evaluate the integral.

**Answer.** Sketch the region of integration (see #31 from Section 14.7 in the text). Then notice the region of integration is

\[
0 \leq z \leq 1, \quad 0 \leq y \leq 1 - z \quad 0 \leq x \leq 1 - y^2.
\]

Therefore,

\[
\int_{0}^{1} \int_{0}^{1-y^2} \int_{0}^{1-y} xyz \, dz \, dx \, dy = \int_{0}^{1} \int_{0}^{1-z} \int_{0}^{1-y^2} xyz \, dx \, dy \, dz
\]

**Promised Formula.** \( \int \int_{R} f(x, y) \, dx \, dy = \int \int_{S} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv. \)