Name: __________________________

Hints and Answers

Instructions. Do the first question, and any four of the last five questions. Please do your best, and show all appropriate work. Thank you.

1. (18 pts) (a) Compute the divergence of the vector field \( \mathbf{F}(x, y, z) = zi + x^2j + y^2k \).

Answer. \( \text{div} \mathbf{F} = 0 + 0 + 0 = 0 \).

(b) Compute curl\( \mathbf{F} \) where \( \mathbf{F} \) is as in (a).

Answer.
\[
\text{curl} \mathbf{F} = \begin{vmatrix}
i & j & k \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
z & x^2 & y^2
\end{vmatrix} = 2yi + 1j + 2xk.
\]

(c) Suppose \( C \) is a closed piecewise smooth simply connected curve that encloses the region \( R \) where \( C \) is oriented in a counterclockwise direction. How is \( \int_C x \, dy - y \, dx \) related to the area of \( R \)?

Answer.
\[
\int_C x \, dy - y \, dx = \int \int_R [1 - (-1)] \, dA = \text{twice the area of } R.
\]

(d) Explain how you would test whether a vector field \( \mathbf{F} = M(x, y, z)i + N(x, y, z)j + P(x, y, z)k \) is conservative when \( M, N \) and \( P \) have continuous first partial derivatives.

Answer. Compute the curl of \( \mathbf{F} \). If \( \text{curl} \mathbf{F} = 0 \), then \( \mathbf{F} \) is conservative; otherwise \( \mathbf{F} \) is not conservative.

(e) Parametrize the curve \( y = x^2 - 2 \) starting at \((3, 7)\) and ending at \((0, -2)\).

Answer. There are many possibilities: \( x = 3 - t \) and \( y = (3 - t)^2 - 2 = t^2 - 2t + 7 \) where \( 0 \leq t \leq 3 \).

(f) Suppose \( f(x, y) = x - y^2 \) is a potential function for \( \mathbf{F}(x, y) \). Find the work done by the force field \( \mathbf{F} \) on a particle moving along the top half of the circle \( x^2 + y^2 = 16 \) in a clockwise direction.

Answer. \[
\int_C \mathbf{F} \cdot \, dr = x - y^2 \bigg|_{(4,0)}^{-(4,0)} = 4 - (4) = 8.
\]
2. (8 pts) (a) Sketch the vector field \( \mathbf{F}(x, y) = 2x \mathbf{i} + y^2 \mathbf{j} \) by sketching some representative vectors.

**Answer.** The sketch is left to the reader.

(b) Is \( \mathbf{F} \) conservative? Justify your answer.

**Answer.** Yes, because \( \frac{\partial}{\partial x}(y^2) = 0 = \frac{\partial}{\partial y}(2x) \).

(c) Find a potential function for \( \mathbf{F} \) or explain why no potential function exists.

**Answer.** A potential function is \( f(x, y) = x^2 + \frac{y^3}{3} \).

3. (8 pts) Use a line integral to find the exact surface area of a retaining wall whose height is given by \( 2x \), and the base of the wall is on the parabola \( y = x^2 \) starting at \( (0, 0) \) and ending at \( (2, 4) \).

**Answer.** Parametrize \( C \) as \( x = t, \ y = t^2 \) where \( 0 \leq t \leq 2 \). Therefore, \( ds = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{1 + 4t^2} \ dt \). Then

\[
\text{S.A.} = \int_C 2x \, ds = \int_0^2 2t \sqrt{1 + 4t^2} \, dt = \int_1^{17} \frac{u^{1/2}}{4} \, du \text{ where } u = 1 + 4t^2, \ du = 8t \, dt = \frac{1}{6}u^{3/2}_{\mid_1}^{17} = \frac{1}{6}(17^{3/2} - 1).
\]
4. (8 pts) Let $C$ be the triangle with vertices $(0,0)$, $(1,-1)$ and $(1,1)$ oriented in a counterclockwise direction. Set-up the integral(s) needed to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = 2y\mathbf{i} - x^2\mathbf{j}$ along $C$. Do not evaluate the integrals, but express the work as a sum of integrals that MathCAD could evaluate.

**Answer.** Starting at the origin and proceeding in a counter clockwise direction, parametrize $C$ as three line segments $C_1$, $C_2$, and $C_3$ where

$C_1$: $x = t$, $y = -t$, $t = 0$ to 1 so $dx = dt$, $dy = -dt$.

$C_2$: $x = 1$, $y = t$, $t = -1$ to 1, so $dx = 0 dt$ and $dy = dt$.

$C_3$: $x = t$, $y = t$, $t = 1$ to 0, so $dx = dt$ and $dy = dt$.

Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C 2y\, dx - x^2\, dy$$

$$= \int_0^1 -2t\, dt - t^2(-dt) + \int_{-1}^1 -1\, dt + \int_0^1 (2t - t^2)\, dt$$

$$= \int_0^1 (t^2 - 2t)\, dt + \int_{-1}^1 -1\, dt + \int_0^1 (t^2 - 2t)\, dt$$

$$= \frac{2t^3}{3} - 2t^2\bigg|_0^1 - 2 = \frac{2}{3} - 2 - 2 = -\frac{10}{3}.$$ 

Note: the question did not require the evaluation, but it was completed to compare the answer with question 5.

5. (8 pts) Use Green’s theorem to calculate the work done by the vector field along the curve $C$ in question 4.

**Answer.**

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_R (-2x - 2)\, dA$$

$$= \int_0^1 \int_{-x}^x (-2x - 2)\, dy\, dx = \int_0^1 \left[ -2xy - 2y \bigg|_{y=-x}^{y=x} \right]\, dx$$

$$= \int_0^1 \left[ -2x^2 - 2x - 2x^2 - 2x \right]\, dx = \int_0^1 \left[ -4x^2 - 4x \right]\, dx$$

$$= \left. -\frac{4}{3}x^3 - 2x^2 \right|_0^1 = -\frac{4}{3} - 2 = -\frac{10}{3}.$$
6. (8 pts) (a) Find a potential function for the vector field
\[ \mathbf{F}(x, y, z) = 2xy \mathbf{i} + (x^2 + z^2) \mathbf{j} + (2yz + 1) \mathbf{k}. \]

**Answer.** By inspection \( f(x, y, z) = x^2y + yz^2 + z \). To check, compute the gradient
\[ \nabla f = (2xy, x^2 + z^2, 2yz + 1) \]
as desired.

(b) Evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} \) is the vector field in (a), and \( C \) is the line segment starting at \((1, 1, 0)\) and ending at \((0, 2, 3)\).

**Answer.**
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = x^2y + yz^2 + z \bigg|_{(1,1,0)}^{(0,2,3)} = (0 + 18 + 3) - (1 + 0 + 0) = 21 - 1 = 20. \]