Recall the polar coordinate basic conversion formulas:

\[ x = r \cos \theta, \quad y = r \sin \theta \quad x^2 + y^2 = r^2, \quad \tan \theta = \frac{y}{x} \quad \text{when} \ x \neq 0 \]

The area of the shaded \( \Delta A \) region in Figure 1 is

\[
\pi (r_2^2 - r_1^2) \frac{(\theta_2 - \theta_1)}{2\pi} = \left( \frac{r_1 + r_2}{2} \right) \Delta r \Delta \theta
\]

where \( \Delta \theta = \theta_2 - \theta_1 \) and \( \Delta r = r_2 - r_1 \). In the limiting case as \( \Delta r \to 0 \) and \( \Delta \theta \to 0 \), we obtain

\[
dA = r \, dr \, d\theta
\]

In general when a region \( R \) is described by \( g(\theta) \leq r \leq h(\theta) \) and \( \alpha \leq \theta \leq \beta \) (see Figure 2) one has

\[
\int_R \int f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta
\]

**Example 1.** Find the volume of the solid that lies below the paraboloid \( z = 9 - x^2 - y^2 \) and above the plane \( z = 0 \).

**Answer.** This area is given by \( \int_R \int (9 - x^2 - y^2) \, dA \) where \( R = \{(x, y) : x^2 + y^2 \leq 9\} \). In polar coordinates, this becomes

\[
V = \int_0^{2\pi} \int_0^3 (9 - r^2) r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left. \frac{9}{2} r^2 - \frac{r^4}{4} \right|_0^3 d\theta
\]

\[
= \int_0^{2\pi} \frac{81}{4} d\theta = \frac{81}{2} \pi
\]
The following example illustrates the value of changing coordinate systems. Both MathCAD and Wolfram Alpha struggled with the following integral (April 29, 2014) as is.

**Example 2.** Evaluate the following integral by converting it to polar coordinates

\[
\int_0^2 \int_y^\sqrt{8-y^2} 8(x^2 + y^2)^{1/2} \, dx \, dy
\]

**Answer.** In polar coordinates the integral becomes (a sketch of the region is useful)

\[
\int_0^2 \int_y^\sqrt{8-y^2} 8(x^2 + y^2)^{1/2} \, dx \, dy = \int_0^{\pi/4} \int_0^{\sqrt{8}} 8r^3 \, r \, dr \, d\theta
\]

\[
= \frac{\pi}{4} \left. 8r^4 \right|_0^{\sqrt{8}} = \frac{8\pi (8)^{1/4}}{3} = \frac{16\pi \sqrt{8}}{3}
\]

\[
\approx 47.39075
\]

Screen shots of what MathCad and Wolfram Alpha did with integral are given below. Both failed to evaluate the integral symbolically, but both provided numerical values close to the above answer.
Example 3. Write the integral \( \int_0^2 \int_0^{\sqrt{2x-x^2}} xy \, dy \, dx \) in polar coordinates, and then evaluate it in the form that looks most convenient to you.

Answer. If \( y = \sqrt{2x-x^2} \), then \( y^2 = 2x - x^2 \) where \( y \geq 0 \) and so \( x^2 - 2x + y^2 = 0 \) ans completing the square
\[
(x-1)^2 + y^2 = 1^2, \quad \text{where} \ y \geq 0
\]
so the region of integration is the top half of the circle of radius 1 centered at (1, 0). In polar form, \( x^2 + y^2 - 2x = 0 \) implies \( r^2 - 2r \cos \theta = 0 \) and so \( r = 0 \) or \( r = 2 \cos \theta \) where \( 0 \leq \theta \leq \pi/2 \) because the region is only the top half of the circle. Thus in polar form, the integral becomes
\[
\int_0^{\pi/2} \int_0^{2 \cos \theta} (r \cos \theta)(r \sin \theta) r \, dr \, d\theta = \int_0^{\pi/2} \int_0^{2 \cos \theta} r^3 \cos \theta \sin \theta \, dr \, d\theta
\]
The integral is relatively easy in both rectangular and polar coordinates, so we evaluate the rectangular integral directly:
\[
\int_0^2 \int_0^{\sqrt{2x-x^2}} xy \, dy \, dx = \int_0^2 \frac{xy^2}{2} \bigg|_0^{\sqrt{2x-x^2}} \, dx = \frac{1}{2} \int_0^2 (2x^2 - x^3) \, dx = \frac{1}{2} \left( \frac{2x^3}{3} - \frac{x^4}{4} \bigg|_0 \right) = \frac{1}{2} \left( \frac{2^4}{3} - \frac{2^4}{4} \right) = \frac{2}{3}
\]
For reference, in polar form, the integral is
\[
\int_0^{\pi/2} \int_0^{2\cos \theta} r^3 \cos \theta \sin \theta \, dr \, d\theta = \int_0^{\pi/2} \frac{r^4}{4} \cos \theta \sin \theta \bigg|_0^{2\cos \theta} \, dr
\]
\[
= \int_0^{\pi/2} \frac{(2^4) \cos^5 \theta \sin \theta}{4} \, d\theta
\]
\[
= \frac{-2^4 \cos^6(\theta)}{24} \bigg|_0^{\pi/2} = \frac{2}{3} [0 - (-1)] = \frac{2}{3}
\]

Example 4. A cylindrical drill of radius 7 units is used to bore a hole through the center of a solid ball of radius 9 units. Use a double integral in polar coordinates to find the volume of the annular shaped solid that remains.

Answer. The volume is given as follows
\[
V = \int_0^{2\pi} \int_0^{9} 2\sqrt{9^2 - r^2} \, r \, dr \, d\theta
\]
\[
= 2\pi \left[ -\frac{2}{3} (81 - r^2)^{3/2} \right]_0^9
\]
\[
= \frac{4\pi}{3} (81 - 49)^{3/2} = \frac{4\pi}{3} (32)^{3/2} \approx 758.25202145 \text{ units}^3
\]
Practice Exercises on Double Integrals in Polar Coordinates

1. Find the volume of the solid region bounded above by the graph

\[ z = \sqrt{36 - x^2 - y^2} \]

and below by the region \( R \) in the \( xy \)-plane given by

\[ x^2 + y^2 \leq 16 \]

2. Convert \( \int_{-4}^{1} \int_{0}^{\sqrt{16-x^2}} 12x \sqrt{x^2 + y^2} \, dy \, dx + \int_{-1}^{0} \int_{\sqrt{1-x^2}}^{\sqrt{16-x^2}} 12x \sqrt{x^2 + y^2} \, dy \, dx \) to a double integral in polar coordinates and evaluate it.

3. A cylindrical drill of radius 4 units is used to bore a hole through the center of a solid ball of radius 9 units. Use a double integral in polar coordinates to find the volume of the annular shaped solid that remains.

4. Find the volume of the solid enclosed by \( z = 8 - x^2 - y^2 \) and \( z = 2 \sqrt{x^2 + y^2} \).

5. Write the integral \( \int_{0}^{8} \int_{0}^{\sqrt{8x-x^2}} xy \, dy \, dx \) in polar coordinates, and then evaluate it in the form that looks most convenient to you.
Practice Exercises on Double Integrals in Polar Coordinates with Solutions.

1. Find the volume of the solid region bounded above by the graph

\[ z = \sqrt{36 - x^2 - y^2} \]

and below by the region \( R \) in the \( xy \)-plane given by

\[ x^2 + y^2 \leq 16 \]

**Solution:** In polar coordinates the region \( R \) is given by \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq r \leq 4 \). We also write

\[ z = \sqrt{36 - (x^2 + y^2)} = \sqrt{36 - r^2} \]

Then the volume is given by

\[
V = \int \int_R f(x,y) \, dA = \int_0^{2\pi} \int_0^4 \sqrt{36 - r^2} \, r \, dr \, d\theta \\
= -\frac{1}{3} \int_0^{2\pi} (36 - r^2)^{3/2} \bigg|_{r=0}^{r=4} d\theta \\
= \frac{2\pi}{3} [(36)^{3/2} - (20)^{3/2}] \\
\]

2. Convert \( \int_{-4}^{0} \int_{\sqrt{16-x^2}}^{\sqrt{16-x^2}} 12x \sqrt{x^2+y^2} \, dy \, dx + \int_{-1}^{0} \int_{\sqrt{16-x^2}}^{\sqrt{16-x^2}} 12x \sqrt{x^2+y^2} \, dy \, dx \) to a double integral in polar coordinates and evaluate it.

**Solution:** A sketch is quite useful for doing the conversion. The polar integral and its evaluation are as follows.

\[
\int_{\frac{\pi}{2}}^{\pi} \int_{1}^{4} 12r^2 \cos \theta \, r \, dr \, d\theta = \int_{\frac{\pi}{2}}^{\pi} 3r^4 \cos \theta \bigg|_{r=1}^{r=4} d\theta \\
= \int_{\frac{\pi}{2}}^{\pi} 3(4^4 - 1^4) \cos \theta \, d\theta \\
= 765 \int_{\frac{\pi}{2}}^{\pi} \cos \theta \, d\theta = 765 \left[ \sin \theta \right]_{\frac{\pi}{2}}^{\pi} \\
= 765[0 - 1] = -765. 
\]
3. A cylindrical drill of radius 4 units is used to bore a hole through the center of a solid ball of radius 9 units. Use a double integral in polar coordinates to find the volume of the annular shaped solid that remains.

**Solution:** The volume is given as follows

\[
V = \int_0^{2\pi} \int_4^9 2\sqrt{9^2 - r^2} \, r \, dr \, d\theta
\]

\[
= 2\pi \left[ -\frac{2}{3} (81 - r^2)^{3/2} \right]_4^9
\]

\[
= \frac{4\pi}{3} (81 - 16)^{3/2} = \frac{4\pi}{3} (65)^{3/2} \approx 2195.12190849 \text{ units}^3
\]

4. Find the volume of the solid enclosed by \( z = 8 - x^2 - y^2 \) and \( z = 2\sqrt{x^2 + y^2} \).

**Solution:** In polar coordinates the surfaces are \( z = 8 - r^2 \) and \( z = 2r \) where \( r \geq 0 \). These curves intersect when \( 8 - r^2 = 2r \) and the only solution where \( r \geq 0 \) is \( r = 2 \). The volume is then given by

\[
V = \int_0^{2\pi} \int_0^2 [8 - r^2 - 2r] \, r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^2 [8r - r^3 - 2r^2] \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left( \frac{8r^2}{2} - \frac{r^4}{4} - \frac{2r^3}{3} \Big|_0^2 \right) \, d\theta
\]

\[
= \int_0^{2\pi} \left( \frac{32}{2} - \frac{16}{4} - \frac{16}{3} \right) \, d\theta
\]

\[
= \frac{80\pi}{6} \approx 41.88790
\]

5. Write the integral \( \int_0^8 \int_0^{\sqrt{8x-x^2}} xy \, dy \, dx \) in polar coordinates, and then evaluate it in the form that looks most convenient to you.
\textbf{Solution:} If \( y = \sqrt{8x - x^2} \), then \( y^2 = 8x - x^2 \) where \( y \geq 0 \) and so \( x^2 - 8x + y^2 = 0 \) ans completing the square
\[
(x - 4)^2 + y^2 = 4^2, \quad \text{where } y \geq 0
\]
so the region of integration is the top half of the circle of radius 4 centered at \((4,0)\). In polar form, \( x^2 + y^2 - 8x = 0 \) implies \( r^2 - 8r \cos \theta = 0 \) and so \( r = 0 \) or \( r = 8 \cos \theta \) where \( 0 \leq \theta \leq \pi/2 \) because the region is only the top half of the circle. Thus in polar form, the integral becomes
\[
\int_{0}^{\pi/2} \int_{0}^{8 \cos \theta} (r \cos \theta)(r \sin \theta) \, r \, dr \, d\theta = \int_{0}^{\pi/2} \int_{0}^{8 \cos \theta} r^3 \cos \theta \sin \theta \, dr \, d\theta
\]
The integral is relatively easy in both rectangular and polar coordinates, so we evaluate it both ways. The rectangular integral is
\[
\int_{0}^{8} \int_{0}^{\sqrt{8x-x^2}} xy \, dy \, dx = \int_{0}^{8} xy^2 \bigg|_{0}^{\sqrt{8x-x^2}} \, dx
\]
\[
= \frac{1}{2} \int_{0}^{8} (8x^2 - x^3) \, dx
\]
\[
= \frac{1}{2} \left( \frac{8x^3}{3} - \frac{x^4}{4} \right)_{0}^{8}
\]
\[
= \frac{1}{2} \left( \frac{8^4}{3} - \frac{8^4}{4} \right) = \frac{8^4}{24} = \frac{512}{3}
\]
In polar form, the integral is
\[
\int_{0}^{\pi/2} \int_{0}^{8 \cos \theta} r^3 \cos \theta \sin \theta \, dr \, d\theta = \int_{0}^{\pi/2} \int_{0}^{8 \cos \theta} \frac{r^4}{4} \cos \theta \sin \theta \bigg|_{0}^{8 \cos \theta} \, dr
\]
\[
= \int_{0}^{\pi/2} \frac{(8^4) \cos^5 \theta \sin \theta}{4} \, d\theta
\]
\[
= -\frac{8^4 \cos^6(\theta)}{24} \bigg|_{0}^{\pi/2} = \frac{512}{3} \left[0 - (-1)\right] = \frac{512}{3}
\]