Instructions: Do the following problems: 5.1.1(1),(2), 5.1.3odd, 5.1.4(1),(4), 5.2.1, 5.2.6, 5.2.11, 5.3.1(1),(2),(4), 5.3.2(1),(2), 5.3.10, 5.3.13(2),(3).

5.1.1 (1) \[3\] = \{-3, 3\}, \[-3\] = \emptyset and \[6\] = \{-6, 6\}.
(2) \[3\] = \{3n : n \in \mathbb{Z}\}, \[-3\] = \{3n : n \in \mathbb{Z}\} and \[6\] = \{6n : n \in \mathbb{Z}\}.

5.1.3 (1) is symmetric but not reflexive or transitive.
(3) is transitive but not reflexive or symmetric.
(5) is transitive and symmetric, but not reflexive.
(7) is reflexive and transitive, but not symmetric.

5.1.4 (1) is not reflexive since \(-3\not S -3\), it is not symmetric since \(-2S2\) but \(2S -2\). However it is transitive since \(xSy\) and \(ySz\) imply \(y = |x|\) and \(z = |y|\) and so \(z = |x|\) which means \(xSz\).
(4) is reflexive, symmetric and transitive.

5.2.1 (1), (4), and (5) are true, while (2) and (3) are false.

5.2.6 (1) We claim that \(a + c \equiv b + c \pmod{n}\) implies \(a \equiv b \pmod{n}\) whenever \(a, b, c \in \mathbb{Z}\).

Proof. Let \(a, b, c \in \mathbb{Z}\). Then, \(a + c \equiv b + c \pmod{n}\) implies \((a + c) - (b + c) = kn\) for some \(k \in \mathbb{Z}\). Thus, \(a - b = kn\) and so \(a \equiv b \pmod{n}\). \(\square\)

(2) It is not true that \(ac \equiv bc \pmod{n}\) implies \(a \equiv b \pmod{n}\) for all \(a, b, c \in \mathbb{Z}\).

Proof. The easiest counterexamples are when \(c = 0\) and \(a\) and \(b\) are not congruent modulo \(n\). A slightly more subtle counterexample is \(6 \cdot 7 \pmod{21} \equiv 3 \cdot 7 \pmod{21}\) as both are equal to 0, while \(6 \not\equiv 3 \pmod{21}\). \(\square\)

5.2.11 We claim that \[\sum_{i=1}^{m} a_i 10^{i-1} \equiv \sum_{i=1}^{m} a_i \pmod{9}\].

In particular, this implies a number is divisible by 9 if and only if the sum of its digits are divisible by 9.
Proof. Using the generalized version of Lemma 5.2.5 we compute

\[
\left( \sum_{i=1}^{m} 10^{i-1}a_i \right) \pmod{9} \equiv \sum_{i=1}^{m} 10^{i-1} \pmod{9} a_i \pmod{9} \\
\equiv \sum_{i=1}^{m} a_i \pmod{9}
\]

since \(1 \equiv 10^k \pmod{9}\) for all \(k = 0, 1, 2, \ldots\).

In particular, the left hand side of the above equation is equal to 0 when the number is divisible by 9, and this occurs if and only if the right hand side of the equation is equal to 0, that is, the sum of the digits is divisible by 9.

5.3.1 (1) The relation \(M\) on \(\mathbb{R}\) given by \(xMy\) iff \(x - y\) is an integer is an equivalence relation.

Proof. First, \(M\) is reflexive since \(x - x = 0\) is an integer for all \(x \in \mathbb{R}\). Next, the relation is symmetric, because if \(xMy\), then \(x - y\) is an integer, and therefore \(y - x\) is an integer, and so \(yMx\). Finally, \(M\) is transitive: if \(xMy\) and \(yMz\), then \(x - y\) and \(y - z\) are integers. It follows that \(x - z = (x - y) + (y - z)\) is an integer. Consequently, \(xMz\).

(2) This is not an equivalence relation. We verify this by showing it is not reflexive. Indeed, \(a \not\sim a\) for \(a < 0\) since \(a \neq |a|\).

(4) This is not an equivalence relation. One can verify that it is not reflexive, because a person is not their own first cousin, one can verify it is not transitive, because a first cousin of a first cousin need not be a first cousin of the original person.

5.3.2 (1) \([0] = \{0\}\) and \([3] = \{-3, 3\}\).

(2) \([0] = \{n\pi : n \in \mathbb{Z}\}\), \([3] = \{3 + 2n\pi, (2n + 1)\pi - 3 : n \in \mathbb{Z}\}\).

5.3.10 (1), (3) and (6) are partitions of \([0, \infty)\) while the others are not. The sets in (2), (4) and (5) are not pairwise disjoint.

5.3.13 (2) The equivalence relation on \(\mathbb{R}\) can be described by \(x \sim y\) if and only if \(|x| = |y|\).

(3) The equivalence relation on \(\mathbb{R}^2\) can be described by \((x, y) \sim (v, w)\) if and only if \(x^2 + y^2 = v^2 + w^2\).