Note: in these solutions we often use the following fact.

If $A \sim C$ and $B \sim D$, then $A \times B \sim C \times D$, where $A \sim C$ means $A$ and $C$ have the same cardinality.

**Proof.** Because $A \sim C$ and $B \sim D$, there are bijections $f : A \to C$ and $g : B \to D$, then $h(a, b) = (f(a), g(b))$ defines a bijection from $A \times B$ to $C \times D$. Indeed, if $(a_1, b_1) \neq (a_2, b_2)$, then either $a_1 \neq a_2$ or $b_1 \neq b_2$ and so either $f(a_1) \neq f(a_2)$ or $g(b_1) \neq g(b_2)$ because $f$ and $g$ are injections. In either case, $h(a_1, b_1) \neq h(a_2, b_2)$, and so $h$ is an injection. To show that $h$ is a surjection, let $(c, d) \in C \times D$, because $f$ and $g$ are surjections, we choose $a \in A$ and $b \in B$ so that $f(a) = c$ and $g(b) = d$. Then $h(a, b) = (f(a), g(b)) = (c, d)$, and so $h$ is a surjection. \(\square\)

1. Find a bijection from $\mathbb{N}$ to all integer powers of 10, i.e, find a bijection

$$f : \mathbb{N} \to \{\ldots, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^3, \ldots\}.$$ 

**Answer:** One possible bijection is $f$ defined by $f(n) = 10^{-\frac{n}{2}}$ if $n$ is even, and $f(n) = 10^{\frac{n-1}{2}}$ when $n$ is odd. The first few values of this mapping are as follows: $f(1) = 10^0 = 1$, $f(2) = 10^{-1}$, $f(3) = 10^1$, $f(4) = 10^{-2}$, $f(5) = 10^2$, \ldots.

2. Construct a bijection from $[0, 2]$ onto $(0, 1)$.

**Answer:** One possible bijection is defined as follows. Define $f(x) = \frac{x}{2}$ if $x \notin \{2^n : n = 1, 0, -1, -2, \ldots\} \cup \{0\}$, let $f(0) = 1/2$, $f(2k) = 2k^{-3}$ for $k = 1, 0, -1, -2, \ldots$.

3. Prove that the cardinality of the transcendental numbers is $c$.

**Proof.** As proved in class, the set of algebraic numbers is countable (see Theorem 6.2.2), so if we let $T$ represent the transcendental numbers, $\mathbb{R} = T \cup \{x_i\}_{i=1}^{\infty}$, where $\{x_i\}_{i=1}^{\infty}$ is an enumeration of the algebraic numbers. Therefore, the transcendental numbers are uncountable. Now fix a countable collection $\{y_i\}_{i=1}^{\infty} \subset T$ (since the transcendental numbers are nonempty, we can let $y_1 = t$ be some transcendental number, then let $y_k = kt$ to get an infinite countable collection of distinct transcendental numbers). Now define a bijection from $\mathbb{R}$ to $T$ by $f(t) = t$ if $t \in \mathbb{R} - (\{x_k\}_{k=1}^{\infty} \cup \{y_k\}_{k=1}^{\infty})$, $f(x_k) = x_{2k}$ if $k \in \mathbb{N}$ and $f(y_k) = x_{2k-1}$. \(\square\)

4. (a) Prove that $\mathbb{R}^n$ and $\mathbb{R}$ have the same cardinality for any $n \in \mathbb{N}$.

(b) Deduce from the case $n = 2$ in (a), that $\mathbb{R}$ and $\mathbb{C}$ have the same cardinality.
Proof. (a) In class we saw that \((0, 1) \times (0, 1) \sim (0, 1)\). According to Exercise 5 below, \(\mathbb{R} \sim (0, 1)\) and so \(\mathbb{R} \times \mathbb{R} \sim (0, 1) \times (0, 1) \sim \mathbb{R}\). Therefore, \(\mathbb{R}^2 \sim \mathbb{R}\), i.e. they have same cardinality, and clearly \(\mathbb{R} \sim \mathbb{R}\). So the statement is valid for \(n = 1\) and \(n = 2\). Now suppose by way of mathematical induction that \(\mathbb{R}^k \sim \mathbb{R}\) for some \(k \geq 2\). Then \(\mathbb{R}^{k+1} = \mathbb{R}^k \times \mathbb{R}\) and now \(\mathbb{R}^k \times \mathbb{R} \sim \mathbb{R} \times \mathbb{R}\) and so \(\mathbb{R}^{k+1} \sim \mathbb{R}^2\). Moreover, \(\mathbb{R}^2 \sim \mathbb{R}\) and so \(\mathbb{R}^{k+1} \sim \mathbb{R}\). By the Principle of Mathematical Induction, \(\mathbb{R}^k \sim \mathbb{R}\) for all \(k \in \mathbb{N}\).

(b) Observe that \(f(a, b) = a + bi\) defines a bijection from \(\mathbb{R}^2 \to \mathbb{C}\). Therefore \(\mathbb{R}^2 \sim \mathbb{C}\). Now \(\mathbb{R} \sim \mathbb{R}^2\) and \(\mathbb{R}^2 \sim \mathbb{C}\) imply \(\mathbb{R} \sim \mathbb{C}\).

5. Write an explicit one-to-one correspondence between \((a, b)\) and \((-\infty, \infty)\).

Answer: One such bijection is \(f(t) = \frac{b - a}{\pi} (\arctan t) + \frac{a + b}{2}\). One can check this is a bijection because we know \(\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})\) is a bijection, and \(g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (a, b)\) defined by \(g(t) = \frac{b - a}{\pi} t + \frac{a + b}{2}\) is a bijection. According to Lemma 4.4.2, \(f\) must be a bijection because it is the composition of two bijections.

6. Suppose that \(A\) and \(B\) have the same cardinality, show that \(\mathcal{P}(A)\) and \(\mathcal{P}(B)\) have the same cardinality.

Proof. Because \(A \sim B\), there is a bijection \(f : A \to B\). Now define a map \(g : \mathcal{P}(A) \to \mathcal{P}(B)\) by \(g(S) = \{b \in B : b = g(s) \text{ for some } s \in S\}\), that is, in the notation of the text \(g(S) = f_*(S)\). Now suppose \(S_1 \neq S_2\) are subsets of \(A\), then \(f_*(S_1) \neq f_*(S_2)\) because \(f\) is one-to-one, so \(g(S_1) \neq g(S_2)\). Therefore \(g\) is an injection. To show that \(g\) is a surjection, let \(T \subseteq B\) be any subset, then let \(S = f_*(T)\). Then \(g(S) = T\) and so \(g\) is onto. Consequently, \(g\) is a bijection from \(\mathcal{P}(A)\) to \(\mathcal{P}(B)\).

7. Use the Schroeder-Berstein theorem to show that any interval that is not a singleton has cardinality \(c\). Hint: if \(I\) is any such interval, then \((a, b) \subseteq I \subseteq \mathbb{R}\) for some \(a < b\).

Proof. Clearly \(f : I \to \mathbb{R}\) defined by \(f(t) = t\) for each \(t \in I\) is an injection from \(I\) into \(\mathbb{R}\). Now, let \((a, b) \subseteq I\) for some \(a < b\). Then, by problem 5, there is an injection \(g : \mathbb{R} \to (a, b)\), thus \(g : \mathbb{R} \to I\) is an injection. By the Schroeder-Bernstein Theorem, \(I\) and \(\mathbb{R}\) have the same cardinality.