Math 415 Assignment 8 (Answers)
Due: Monday, November 10

Note: in these solutions we often use the following fact.

If $A \sim C$ and $B \sim D$, then $A \times B \sim C \times D$, where $A \sim C$ means A and C have the same cardinality.

Proof. Because $A \sim C$ and $B \sim D$, there are bijections $f : A \to C$ and $g : B \to D$, then $h(a, b) = (f(a), g(b))$ defines a bijection from $A \times B$ to $C \times D$. Indeed, if $(a_1, b_1) \neq (a_2, b_2)$, then either $a_1 \neq a_2$ or $b_1 \neq b_2$ and so either $f(a_1) \neq f(a_2)$ or $g(b_1) \neq g(b_2)$ because f and g are injections. In either case, $h(a_1, b_1) \neq h(a_2, b_2)$, and so h is an injection. To show that h is a surjection, let $(c, d) \in C \times D$, because f and g are surjections, we choose $a \in A$ and $b \in B$ so that $f(a) = c$ and $g(b) = d$. Then $h(a, b) = (f(a), g(b)) = (c, d)$, and so h is a surjection. $\square$

1. Find a bijection from $\mathbb{N}$ to all integer powers of 10, i.e, find a bijection

$$f : \mathbb{N} \to \{\ldots, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^3, \ldots\}.$$

Answer: One possible bijection is f defined by $f(n) = 10^{-\frac{n}{2}}$ if n is even, and $f(n) = 10^{\frac{n-1}{2}}$ when n is odd. The first few values of this mapping are as follows: $f(1) = 10^0 = 1$, $f(2) = 10^{-1}$, $f(3) = 10^1$, $f(4) = 10^{-2}$, $f(5) = 10^2$, ...

2. Construct a bijection from $[0, 2]$ onto $(0, 1)$.

Answer: One possible bijection is defined as follows. Define $f(x) = \frac{x}{2}$ if $x \notin \{2^n : n = 1, 0, -1, -2, \ldots\} \cup \{0\}$, let $f(0) = 1/2$, $f(2^k) = 2^{k-3}$ for $k = 1, 0, -1, -2, \ldots$.

3. Prove that the cardinality of the transcendental numbers is c.

Proof. As proved in class, the set of algebraic numbers is countable (see Theorem 6.2.2), so if we let $\mathcal{T}$ represent the transcendental numbers, $\mathbb{R} = \mathcal{T} \cup \{x_i\}_{i=1}^{\infty}$ where $\{x_i\}_{i=1}^{\infty}$ is an enumeration of the algebraic numbers. Therefore, the transcendental numbers are uncountable. Now fix a countable collection $\{y_i\}_{i=1}^{\infty} \subset \mathcal{T}$ (since the transcendental numbers are nonempty, we can let $y_1 = t$ be some transendental number, then let $y_k = kt$ to get an infinite countable collection of distinct transcendental numbers). Now define a bijection from $\mathbb{R}$ to $\mathcal{T}$ by $f(t) = t$ if $t \in \mathbb{R} - (\{x_k\}_{k=1}^{\infty} \cup \{y_k\}_{k=1}^{\infty})$, $f(x_k) = x_{2k}$ if $k \in \mathbb{N}$ and $f(y_k) = x_{2k-1}$. $\square$

4. (a) Prove that $\mathbb{R}^n$ and $\mathbb{R}$ have the same cardinality for any $n \in \mathbb{N}$.

(b) Deduce from the case $n = 2$ in (a), that $\mathbb{R}$ and $\mathbb{C}$ have the same cardinality.

Proof. (a) In class we saw that $(0,1)\times(0,1) \sim (0,1)$. According to Exercise 5 below, $\mathbb{R} \sim (0,1)$ and so $\mathbb{R}\times\mathbb{R} \sim (0,1)\times(0,1) \sim (0,1) \sim \mathbb{R}$. Therefore, $\mathbb{R}^2 \sim \mathbb{R}$, i.e. they have same cardinality, and clearly $\mathbb{R} \sim \mathbb{R}$. So the statement is valid for $n = 1$ and $n = 2$. Now suppose by way of mathematical induction that $\mathbb{R}^k \sim \mathbb{R}$ for some $k \geq 2$. Then $\mathbb{R}^{k+1} = \mathbb{R}^k \times \mathbb{R}$ and now $R^k \times \mathbb{R} \sim \mathbb{R} \times \mathbb{R}$ and so $\mathbb{R}^{k+1} \sim \mathbb{R}^2$. Moreover, $\mathbb{R}^2 \sim \mathbb{R}$ and so $\mathbb{R}^{k+1} \sim \mathbb{R}$. By the Principle of Mathematical Induction, $\mathbb{R}^k \sim \mathbb{R}$ for all $k \in \mathbb{N}$.

(b) Observe that $f(a, b) = a + bi$ defines a bijection from $\mathbb{R}^2 \to \mathbb{C}$. Therefore $\mathbb{R}^2 \sim \mathbb{C}$. Now $\mathbb{R} \sim \mathbb{R}^2$ and $\mathbb{R}^2 \sim \mathbb{C}$ imply $\mathbb{R} \sim \mathbb{C}$. □

5. Write an explicit one-to-one correspondence between (a, b) and $(-\infty, \infty)$.

Answer: One such bijection is $f(t) = \dfrac{b-a}{\pi}(\arctan t) + \dfrac{a+b}{2}$. One can check this is a bijection because we know $\arctan : \mathbb{R} \to (-\dfrac{\pi}{2}, \dfrac{\pi}{2})$ is a bijection, and $g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (a, b)$ defined by $g(t) = \dfrac{b-a}{\pi}t + \dfrac{a+b}{2}$ is a bijection. According to Lemma 4.4.2, f must be a bijection because it is the composition of two bijections.

6. Suppose that A and B have the same cardinality, show that $\mathcal{P}(A)$ and $\mathcal{P}(B)$ have the same cardinality.

Proof. Because $A \sim B$, there is a bijection $f : A \to B$. Now define a map $g : \mathcal{P}(A) \to \mathcal{P}(B)$ by $g(S) = \{b \in B : b = g(s) \text{ for some } s \in S\}$, that is, in the notation of the text $g(S) = f_*(S)$. Now suppose $S_1 \neq S_2$ are subsets of A, then $f_*(S_1) \neq f_*(S_2)$ because f is one-to-one, so $g(S_1) \neq g(S_2)$. Therefore g is an injection. To show that g is a surjection, let $T \subseteq B$ be any subset, then let $S = f^*(T)$. Then $g(S) = T$ and so g is onto. Consequently, g is a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. □

7. Use the Schroeder-Berstein theorem to show that any interval that is not a singleton has cardinality c. Hint: if I is any such interval, then $(a, b) \subseteq I \subseteq \mathbb{R}$ for some $a < b$.

Proof. Clearly $f : I \to \mathbb{R}$ defined by $f(t) = t$ for each $t \in I$ is an injection from I into $\mathbb{R}$. Now, let $(a, b) \subseteq I$ for some $a < b$. Then, by problem 5, there is an injection $g : \mathbb{R} \to (a, b)$, thus $g : \mathbb{R} \to I$ is an injection. By the Schroeder-Bernstein Theorem, I and $\mathbb{R}$ have the same cardinality. □