Instructions: Do all five of the following problems. Please do your best, and show all appropriate details in your solutions.

1. Suppose $f : A \rightarrow B$ is a function.
   (a) Define what is meant by the pre-image $f^*(V)$.
   (b) Define what is meant by the image $f_*(U)$.
   (c) For a collection of subsets $\{V_j\}_{j \in J}$ of $B$, prove that $f^* \bigcap_{j \in J} V_j = \bigcap_{j \in J} f^*(V_j)$.
   (d) Is it true that $f_*(U_1 \cap U_2) = f_*(U_1) \cap f_*(U_2)$? Prove or provide a counterexample.

2. (a) Define the terms injection, surjection and bijection.
   (b) Suppose that both $f : A \rightarrow C$ and $g : B \rightarrow D$ are bijections. Show that the function $h : A \times B \rightarrow C \times D$ defined by $h(a, b) = (f(a), g(b))$ is a bijection.
   (c) Find the inverse function of $f : (-\infty, 0] \rightarrow [1, \infty)$ of $f(x) = x^2 + 1$. Be sure to state the domain and range of the inverse function, and to verify it is the inverse function of $f$.

3. (a) Define what is meant by an equivalence relation on a set $A$.
   (b) Define the relation $\sim$ on $\mathbb{R}^2$ by $(a, b) \sim (c, d)$ iff $a^2 + b^2 = c^2 + d^2$. Is $\sim$ an equivalence relation? Verify your assertion.
   (c) Do the relation classes from the relation in (b) form a partition of $\mathbb{R}^2$? If not, explain which properties of a partition are violated. If so, describe the partition, and explain why it is a partition.

4. (a) Find a bijection from the set $S = \{1, 4, 9, 16, 25, \ldots\}$ of all squares of natural numbers onto $\mathbb{Z}$.
   (b) Find a bijection from $(0, 1) \rightarrow (a, b)$ where $a < b$.
   (c) Explain carefully why the irrational numbers in any interval $(a, b)$ with $a < b$ are uncountable.

5. (a) Draw a lattice diagram for $(\mathcal{P}(A), \subseteq)$ where $A = \{a, b, c, d\}$.
   (b) Explain why $(\mathcal{P}(A), \subseteq)$ is not a totally ordered set.
   (c) Let $(S, \leq)$ be a partially ordered set. What properties must $\leq$ have? Be specific.
   (d) Let $(S, \leq)$ be any partially ordered set. Prove that if $S$ has a greatest element, then it is unique.