CHAPTER ONE

Fun and Games

An Introduction to Rigorous Thought

1.1 Silly Stories, Each with a Moral

1.2 Nudges

1.3 The Punch Lines

1.4 From Play to Power
Fun and games, rigorous thought: Here we will discover that in mathematics these go hand in hand. Who says that profound ideas and important insights come only from hard work? Sure, we can consider the discipline of mathematics broadly, from a philosophical perspective, and we can be intrigued by its mysterious wonders. But when we get right down to it, we think mathematics is just plain fun. We hope that one day you will, too.

We start with two fundamental observations:

1. Mathematics involves logical and creative thinking.
2. Thinking can be fun.

By grappling with conundrums serious or otherwise, we can discover significant concepts. As we grope for solutions to silly stories, we begin to develop effective strategies for serious thinking.
1.1 Silly Stories, Each with a Moral

Conundrums That Evoke Techniques of Effective Thinking

Beware gentle knight, there is no greater monster than reason.

MIGUEL DE CERVANTES

You now have a mission. Your mission, should you decide to accept it, is to read the following stories and attempt to answer the questions they raise. The only rules are as follows:

1. Make an earnest attempt to solve each puzzle.
2. Be creative.
3. Don’t give up: If you get stuck, look at the story in a different way.
4. If you become frustrated, stop working, move on, and then return to the story later.
5. Share these stories with your family and friends.
6. HAVE FUN!

A journey of one thousand miles begins with a single step.

CHINESE PROVERB
How do we approach issues in life that need to be resolved? A critical step is simply to begin. Think a bit and then move forward. Taking that first step, though essential, is often scary; more often than not, we do not possess a clear understanding of a complete solution or even see how a solution will eventually fall into place. This situation is like being asked to walk through a forest in the dark. Without knowing the terrain, the natural tendency is to freeze like a deer in headlights. However, we must learn not to let this understandable fear paralyze us intellectually; we must take a step. It is only by stumbling through many small intellectual steps that we are eventually able to make any progress at all.

For example, imagine we’re soccer players with the ball at midfield. In this position we can’t possibly know how a goal will be achieved, and we can’t stop to envision the entire progression of the future before kicking the ball. Instead, we move with the understanding that the specific goal strategy will become clear as opportunities arise.

Just try out ideas with these stories—loosen up, try to kick the ball, and don’t worry if you miss. Remember:

Truth comes out of error more easily than out of confusion.

FRANCIS BACON

After you have given considerable thought to a story, move to the corresponding part in Section 1.2, “Nudges,” where leading questions and suggestions provide a gentle push in the right direction, in case you need a hint. There we also identify some strategies for tackling both mathematical questions and, more importantly, questions that will arise in your life.

Section 1.3, “The Punch Lines,” provides solutions and commentary about how the questions and their resolutions fit into the mathematical landscape. As you think about these stories, you will discover some profound ideas that capture the essence of some deep and beautiful mathematical concepts.

As you proceed, remember the rules on page 4, especially rule 6.

Story 1. That’s a Meanie Genie
On an archeological dig near the highlands of Tibet, Alley discovered an ancient oil lamp. Just for laughs she rubbed the lamp. She quickly stopped laughing when a huge puff of magenta smoke spouted from the lamp, and an ornery genie named Murray appeared. Murray, looking at the stunned Alley, exclaimed, “Well, what are you staring at? Okay, okay,
you’ve found me; you get your three wishes. So, what will they be?” Alley, although in shock, realized she had an incredible opportunity. Thinking quickly, she said, “I’d like to find the Rama Nujan, the jewel that was first discovered by Hardy the High Lama.” “You got it,” replied Murray, and instantly nine identical-looking stones appeared. Alley looked at the stones and was unable to differentiate any one from the others.

Finally she said to Murray, “So where is the Rama Nujan?” Murray explained, “It is embedded in one of these stones. You said you wished to find it. So now you get to find it. Oh, by the way, you may take only one of the stones with you, so choose wisely!” “But they look identical to me. How will I know which one has the Rama Nujan in it?” Alley questioned. “Well, eight of the stones weigh the same, but the stone containing the jewel weighs slightly more than the others,” Murray responded with a devilish grin.

Alley, becoming annoyed, whispered under her breath, “Gee, I wish I had a balance scale.” Suddenly a balance scale appeared. “That was wish two!” declared Murray, “Hey, that’s not fair!” Alley cried, “You want to talk fair? You think it’s fair to be locked in a lamp for 1729 years? You know you can’t get cable TV in there, and there’s no room for a satellite dish! So don’t talk to me about fair,” Murray exclaimed. Realizing he had gone a bit overboard, Murray proclaimed, “Hey, I want to help you out, so let me give you a tip: That balance scale may be used only once.” “What? Only once?” she said, thinking out loud. “I wish I had another balance scale.” ZAP! Another scale appeared. “Okay, kiddo, that was wish three,” Murray snickered. “Hey, just one minute,” Alley said, now regretting not having asked for one million dollars or something more standard. “Well at least this new scale works correctly, right?” “Sure, just like the other one. You may use it only once.” “Why?” Alley inquired. “Because it is a ‘wished’ balance scale,” he said, “so the rule is ‘one scale, one balancing’; it’s just like the rule against using one wish to wish for a hundred more wishes.” “You are a very obnoxious genie.” “Hey, I don’t make up the rules, lady, I just follow them,” he said.

So, Alley may use each of the two balance scales exactly once. Is it possible for Alley to select the slightly heavier stone containing the Rama Nujan from among the nine identical-looking stones? Explain why or why not.

**Story 2. Damsel in Distress**

Long ago, knights in shining armor battled dragons and rescued damsels in distress on a daily basis. Although it is not often stressed in many
stories of chivalry, the rescue often involved logical thinking and creative problem solving by the damsel. Here then is a typical knightly encounter.

Once upon a time, a notorious knight captured a damsel and imprisoned her in a castle surrounded by a square moat that was infested with extraordinarily hungry alligators. The moat was 20 feet across, and no drawbridge existed because after depositing the damsel in the castle, the evil knight had taken it with him (giving his horse one major hernia).

After a time, a good knight rode up and said, “Hail, sweet damsel, for I am here, and thou art there. Now what are we going to do?”

The knight, though good, was not too bright and consequently paced back and forth along the moat looking anxiously at the alligators and trying feebly to think of a plan. While doing so, he stumbled upon two sturdy beams of wood suitable for walking across but lacking sufficient length. Alas, the moat was 20 feet across, but the beams were each only 19 feet long and 8 inches wide. He tried to stretch them and then tried to think. Neither effort proved successful. He had no nails, screws, saws, Superglue, or any other method of joining the two beams to extend their length.

What to do? What to do? Fortunately, the damsel, after a suitable time to allow the good knight to attempt to solve the puzzle on his own, called to the knight and gave him a few hints that enabled him to rescue her. What was the maiden’s suggestion?

This story from medieval times foreshadows our journey into the geometric and the visual.

**Story 3. The Fountain of Knowledge**

During an incredibly elaborate hazing stunt during pledge week, Trey Sheik suddenly found himself alone in the Sahara Desert. His desire to become a fraternity brother was now overshadowed by his desire to find something to drink (these desires, of course, are not unrelated). As he wandered aimlessly through the desert sands, he began to regret his involvement in the whole frat scene. Both hours and miles had passed and Trey was near dehydration. Only now did Trey appreciate the advantages of sobriety. Suddenly, he came upon an oasis.

There, sitting in a shaded kiosk beside a small pool of mango nectar, was an old man named Al Donte. Big Al not only ran the mango bar but was also a travel agent and could book Trey on a two-humped camel back to Michigan. At the moment, however, Trey desired nothing but a large drink of that beautifully translucent and refreshing mangoade. Al informed Trey that he sold the juice only in 8-ounce servings.
and the cost for one serving was $3.50. Trey frantically searched his pockets, and though he found much sand, he also discovered that he had exactly $3.50.

Trey’s jubilation at the thought of liquid coating his parched throat was quickly shattered when Al casually announced that he did not have an 8-ounce glass; all he had was a 6-ounce glass and a 10-ounce glass—neither of which had any markings on it. Al, being a man of his word, would not hear of selling any more or any less than an 8-ounce serving of his libation. Trey, in desperation, wondered whether it was possible to use only the unmarked 6- and 10-ounce glasses to produce exactly 8 ounces in the 10-ounce glass. Do you think it’s possible? If so, explain how, and if not, explain why. This pledge-week prank does whet our appetites for a world of numbers.

**Story 4. Dropping Trou**

Before reading on, remember that truth is sometimes stranger than fiction. The highlight of Professor Burger’s April 1993 talk to more than 300 Williams College students and their parents occurred when, after removing his shoes, he tied his feet together with a stout rope, leaped onto the table, dramatically removed his belt, unzipped his zipper, and dropped his pants. The purple cows (Williams mascots) mooing about on his baggy boxer shorts completed an image not soon forgotten in the annals of mathematical talks. The more conservative parents in the audience were contemplating transferring their sons and daughters to a less “progressive” school.

But then, at the moment of maximum shock and bewilderment, Professor Burger performed the seemingly impossible feat of rehabilitating his fast-sinking reputation. Without removing the rope attached to his feet, he turned his pants inside out and pulled his trousers back to their accustomed position (though now inside out). Thus he simultaneously restored his modesty and his credibility by demonstrating the mathematical triumph of reversing his pants without removing the rope that was tying his feet together.

Please attempt to duplicate Professor Burger’s amazing feat—in the privacy of your room, of course. You will need a rope or cord about 5 feet long. One end of the rope should be tied snugly around one ankle and the other end tied equally snugly about the other ankle. Now, without removing the rope, try to take your pants off, turn them inside out, and put them back on so that you, the rope, and your pants are all exactly as they were at the start, with the exception of your pants being inside out. While
some may find this experiment intriguing, others may find it in poor taste. Everyone will agree, however, that surprising outcomes arise when we bend and contort objects and space.

**Story 5. Dodgeball**

Dodgeball is a game for two players—Player One and Player Two (although any two people can play it, even if they are not named “Player One” and “Player Two”). Each player has a special game board (shown below) and is given six turns.

Player One begins by filling in the first horizontal row of his game board with a run of X’s and O’s. That is, on the first line of his board, he will write either an X or an O in each box. Then Player Two places either an X or an O in the first box of her board. So at this point, Player One has filled in the first row of his board with six letters, and Player Two has filled in the first box of her board with one letter.

The game continues with Player One writing down either an X or an O in each box of the second horizontal row of his board. Then Player Two writes one letter (an X or an O) in the second box of her board. The game proceeds in this fashion until all of Player One’s boxes are filled with X’s and O’s; thus, Player One has produced six rows of six marks each, and Player Two has produced one row of six marks. All marks are visible to both players at all times. Player One wins if any of his rows
exactly matches Player Two’s row (Player One matches Player Two). Player Two wins if her row does not match any of Player One’s rows (Player Two dodges Player One).

Would you rather be Player One or Player Two? Who has the advantage? Can you devise a strategy for either side that will always result in victory? This little game holds within it the key to understanding the sizes of infinity.

**Story 6. A Tight Weave**

Sir Pinsky, a famous name in carpets, has a worldwide reputation for pushing the limits of the art of floor covering. The fashion world stands agog at the clean lines and uncanny coherence of his purple and gold creations. Some call him square because his designs so richly employ that quaint quadrilateral. But squares in the hands of a master can create textures beyond the weavers’ world, although not beyond human imagination.

One day Sir Pinsky began a creation with, as always, a perfect, purple square. However, one square seemed too plain, so in the exact center of it he added a gold square. He saw that the central square implicitly defined eight purple squares surrounding it. As he pondered, he realized that those eight purple squares were identical to his original large square except for two things: (1) Each was one-third the size of the whole square; and (2) none of them had a gold square in its center.

He wondered whether he could further modify his design so that each of the eight small squares would replicate the entire design except for being one-third its size. After much thought, he solved this puzzle and created a design with which his name is associated. Can you sketch and describe his design? Create this design in stages, adding more gold squares at each stage.

Suppose the original square rug is 1 yard by 1 yard. How much gold material would be needed for the second stage? How much for the third stage? Continue computing the area of the gold squares at various stages of the process, and then guess how much gold material will be needed to create the final floor covering. The answer is surprising.

Though our carpet designer is thoroughly modern in all ways, the source of his inspiration is ancient. In Chapter 6, “Fractals and Chaos,”
we will see an example of this style in the 19th-century Buddhist tapestry, *Vaishravana Mandala*.

**Story 7. Let’s Make a Deal**

“Let’s make a deal!” Monty Hall enthuses to the gentleman dressed as a giant singing raisin. The gleeful raisin, whose name is Warren Piece, is ready to wheel and deal as Monty Hall explains the game. “Behind one of these three doors is the Cadillac of your dreams. It is as long as a train and comes complete with a Jacuzzi. Of course, if you spend too much time in the Jacuzzi, your skin will wrinkle, but hey, you’re a raisin, your skin’s already wrinkled.” Monty Hall continues by warning that, “Behind the other doors, however, are two other modes of transportation: two old pack mules. They don’t come with Jacuzzis, although given their exotic odor, you may want to give them a bath.” Of course, the crowd is laughing and applauding, just as the studio sign instructs.

Monty sums it up: “So, there are three closed doors. Behind one is a luxurious car, and behind the other two are mules. Now comes the moment of truth. What door do you pick?” The audience erupts, “Take Door Number 1, take Door Number 1!!” “Door Number 2, Door Number 2!!” “Door Number 3’s the one. Choose 3.” Poor Warren Piece looks around at the crowd, confused and nervous. He considers Door Number 1, then 2, then 3. Finally Monty prompts, “Okay, Warren, which do you want?”

The raisin-clad Warren shouts, “Okay, okay, I’ll take Door Number 3, Door Number 3.” As Monty Hall quiets the overly excited audience, he tells Warren, “I’ll tell you what I’m going to do. I’m going to show you what’s behind one of the doors you didn’t pick. Let’s take a look at what’s behind Door Number 2.” With that, Monty Hall turns to the Vanna White of the 1960s and says, “Please show us what is behind Door Number 2.” The door dramatically swings open, the audience erupts, and Warren breathes once more—behind Door Number 2 is a mule! Monty, knowing where the mules are, always opens one of the mule doors first.

Monty continues, “We now see that the Cadillac is *not* behind Door Number 2. You guessed Door Number 3. I’ll tell you what I’m going to do. If you want, I’ll let you change your mind and choose Door Number 1 instead. It’s up to you. Do you want to stick to your original choice, or do you want to switch?” The audience goes nuts. “Stick, stick,” yell half. “Switch, switch,” advise the others. What to do, what to do?
We now invite you to add your voice to the cacophony—although you need not shout. What should Warren Piece do? Should he switch choices, stick to his original guess, or does it not matter? Here a classic TV game show raises the question: How can we accurately measure the uncertain?

**Story 8. Rolling Around in Vegas**

Recently the swaggering burly billionaire, Mr. Bones, introduced an exciting new dice game at his glitzy High-Rollin’ Bones’ Hotel and Casino. An oversized gold bowl containing four dice is presented to the player. The player inspects each die, removes whichever die seems the luckiest, and throws a $100 chip in the bowl. Then Mr. Bones chooses one of the three remaining dice, takes a $100 chip (picturing his likeness) from his personal collection, and modestly places it into the bowl. Next the player and Mr. Bones roll their respective dice. Whoever rolls the higher number wins the two chips. Simple.

To make the game interesting, the four dice are not the run-of-the-mill dice we remember from the gambling-free days of our youth. While each die does have six sides as usual, their faces are marked in unusual ways. The kit that accompanies each new copy of *The Heart of Mathematics: An invitation to effective thinking* contains these four special dice. Roll ’em on out of your kit.

One die has two 6’s and four 2’s. Another has three 5’s and three 1’s. The third has four 4’s and two blank faces. The last die has 3’s on each face. The dice are not weighted—that is, any face is just as likely to land face-up as any other.

Deep Pockets Drew strides up to the bowl to choose the winning die. Which die should Drew draw? Drew considers the die that has all 3’s. Which die could Mr. Bones select that will beat the all-3’s die two-thirds of the time? After finding that die, we know that the all-3’s die would not be a particularly wise choice.
Next Deep Pockets Drew considers the die with four 4’s and two blank faces. Why will the die with three 5’s and three 1’s beat it two-thirds of the time? After verifying this dicey dominance, we know that selecting the die with four 4’s and two 0’s would not be a smart move.

Drew next considers the die with three 5’s and three 1’s. Why will the die with two 6’s and four 2’s beat it two-thirds of the time? After confirming this superiority, we know that the die with three 5’s and three 1’s would not be the best die.

Only one possibility remains: the die with two 6’s and four 2’s. Is there a die that will beat it two-thirds of the time? Your surprising discovery will show that none of the four dice is the “best” one to select, because each one can be beaten by one of the other three dice two-thirds of the time. Amazing.

So now Drew can put the dice in a circular order where each one beats its clockwise neighbor two-thirds of the time. What is that order? After doing the math, Deep Pockets Drew chooses not to play, and as a result his pockets become deeper.

This intriguing dice game surprisingly leads to the seemingly unrelated insight that the idea of a fair and democratic voting system is impossible—so much for “a government of the people, by the people, and for the people.”

**Story 9. Watsamattawith U?**

Watsamattawith University (WU) is a fine institution, but a paradoxical place. They have comfortable dorm rooms, yet all the students sleep in class; their track team streaks from place to place, yet their cheeks are red with embarrassment as they lose every meet; every student is vegetarian, yet their dining facility is named Holstein Hall; and their student senate is called the House of Representatives. Go figure!

And go figure, indeed—for that is exactly what the registrar of WU did in computing the average GPA of the current graduating class. Every year she computes the average GPA of the male students, the female students, and then all the students from that graduating class.

This year she noticed something most peculiar. The average GPA of the male students in the current graduating class was higher than the average GPA of the male students from last year’s graduating class and the average GPA of the female students in the current graduating class was higher than the average GPA of the female students from last year’s graduating class. Sounds great for the current graduates. Unfortunately, she discovered that the average GPA of the entire graduating class was actually lower than the average GPA of last year’s class. What!??
Given that there were no errors in the registrar’s computations, is it possible that such a phenomenon could occur or is this scenario so ridiculously impossible that merely asking the question deserves the response: *Watsamattawith U?!* If such a scenario is possible, explain how by describing an example where the GPAs of the males goes up, the GPA of the females goes up, but the GPA of all the students goes down. Otherwise respond with . . . well, you know.

**Story 10. Dot of Fortune**

One day three college students were selected at random from the studio audience to play the ever-popular TV game show, “Dot of Fortune.” One of the students had already discovered the power and beauty of mathematical thinking, while the other two were not nearly so fortunate. The stage contained no mirrors, reflective surfaces, or television monitors. The three students were seated around a small round table and blindfolded. As Pat, the host, explained the rules of the game, Vanna affixed a conspicuous but small colored dot to each student’s forehead.

“So, contestants,” Pat explained, “at the sound of the bell you will remove your blindfolds. You will see your two companions sitting quietly at the table, each with a dot on his or her forehead. Each dot is either red
or white. You cannot, of course, see the dot on your own forehead. After you have observed the dots on your companions’ foreheads, you will raise your hand if you see at least one red dot. If you do not see a red dot, you will keep your hands on the table. The object of the game is to deduce the color of your own dot. As soon as you know the color of your dot, hit the buzzer in front of you. Do you understand the rules of the game?”

All the students understood the rules, although the math fan understood them better.

“Are you ready?” asked Vanna after affixing a red dot to each student’s forehead. After the contestants nodded, Vanna rang the bell and they removed their blindfolds. The studio audience quivered with anticipation. The students looked at one another’s dots, and all raised their hands. After some time, the math fan hit her buzzer, knowing what color dot she had. Explain how she knew this. Why did the other students not know? This game requires creative logical reasoning—a powerful means to make discoveries whether they are in math, in life, or even (although rarely) on prime-time TV.
1.2 Nudges

Leading Questions and Hints for Resolving the Stories

When we cannot use the compass of mathematics or the torch of experience . . . it is certain we cannot take a single step forward.

VOLTAIRE

Story 1. That’s a Meanie Genie

Initially, we might think that finding the jewel is impossible because Alley is allowed to make only two comparisons. Instead of comparing stones individually, perhaps she should compare one collection of stones with another collection of stones. Now suppose Alley compares one group with another using the first scale. What can she conclude? What should she do next?

Story 2. Damsel in Distress

Thinking about variations on a situation can shed light on which features are essential and which are not. In this case we might consider a variation in which the damsel in distress is on the other side of a 20-foot river rather than surrounded by a square moat. Unfortunately for the maiden, if she were separated from bliss by a river, she would go blissless, because the two 19-foot beams, in the absence of tools, would still not enable the knight to
Looking at extremes is a potent technique of analysis in many situations and may be helpful here. The extremes, either geometrical ones as in this situation or conceptual ones in other situations, frequently reveal features we might have otherwise overlooked.

**Story 3. The Fountain of Knowledge**

To solve this puzzle, combine trial and error with careful observation. As we observe the outcomes of various attempts, we can teach ourselves what may be possible. Try filling up the 10-ounce glass, and then use it to fill the 6-ounce glass. What do you have now—anything new?

**Story 4. Dropping Trou**

We hope that you physically attempt this exercise. By actually trying a task on your own, it’s often possible to discover insights that otherwise may have been hidden from view (particularly in this case).

You will notice that the rope does restrict the amount of movement of your pants. Your mission is to discover means to work around such restrictions.
constraints. For example, try moving parts of the pants through other parts. You may first want to try this task wearing shorts rather than long pants.

**Story 5. Dodgeball**

Play this game a few times with a friend. Switch roles so that each of you has the opportunity to be Player One and Player Two. Remember, if you are Player One, your goal is to match one of your rows with your opponent’s row. If you are Player Two, you want to dodge all six of your opponent’s rows; that is, you want your row to differ in at least one spot from each of the six rows of your opponent. Who would you rather be: Player One or Player Two?

**Story 6. A Tight Weave**

Consider a purple square that has a smaller gold square in its center. How do each of the eight surrounding squares differ from the whole picture? They are much the same except that the whole picture has a gold square in the middle, and each of the eight surrounding squares is solid purple. How could you modify those eight one-third-size surrounding squares to make them look like smaller copies of the entire picture?

Now let’s ask the question again: “In the picture you now have, is each of the eight one-third-size squares identical to smaller copies of the whole picture?” No. How would you modify each one-third-size square-with-a-gold-center to make it identical to the whole new figure? Are you done?

Draw several steps of this repetitive process. At each stage, add up the areas of all the gold squares. When should you stop this process?

**Story 7. Let’s Make a Deal**

Suppose the raisin’s initial guess was wrong. What would be the result if he were to change his answer?

Suppose instead of three doors, there were ten doors. After Warren Piece guessed Door Number 3, suppose Monty Hall opened eight of the remaining doors and all had mules. Should our raisin switch in that game? Why?
Story 8. Rolling Around in Vegas
To compare two dice, consider making a chart with the columns labeled by the six numbers on one die (some of them duplicates) and the rows labeled by the six numbers of the other die (some of them duplicates). Then put the corresponding numbers in each square and see in how many squares one die beats the other. If one die wins 24 of the 36 times, then that die will win two-thirds of the time. A chart for the die with three 5’s and three 1’s versus the die with four 4’s and two blanks has been started for you.

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Story 9. Watsamattawith U?
The natural initial reaction to the question is *Watsamattawith U?!* However, statistical issues often can be both subtle and counterintuitive. Suppose there are 300 students in each of the graduating classes. In last year’s class, half were male and half were female. Suppose that the average GPA of the men was 2.0 while the average GPA of the women was 3.5. Then the average GPA for last year’s graduating class was 2.75—which is halfway between 2.0 and 3.5. Can you create a graduating class of 300 students for which the average GPA of the men is 2.1, the average GPA of the women is 3.6, and yet the average GPA of the entire graduating class descended to 2.65? Bonus “nudge”: Answer—*Yes, you can!*

Story 10. Dot of Fortune
Sometimes no action is action enough. Put yourself in the position of one of the three contestants. You know that the dot on your forehead is either red or white. The trick to figuring out this conundrum is to imagine what would happen if you were wearing a white dot. You are sitting at the table looking at two red dots. What would your two companions be seeing? What could they deduce? What would they do? What did they do? What can you conclude?

**STOP!**
- Do not proceed to the next section until you have thought about the stories, read the previous section, and tried to come up with answers. No peeking!
1.3 The Punch Lines

Solutions and Further Commentary

Mathematics seems to endow one with something like a new sense.

CHARLES DARWIN

Story 1. That’s a Meanie Genie

Alley identifies which stone contains the jewel with no problem because she has read *The Heart of Mathematics*. She arranges the stones into three groups of three and places one group on one side of the first balance scale and another group on the other side. What can she conclude? If both sides weigh the same, then she knows that the (heavier) jewel must be in the third group of three. If, however, one side is heavier than the other, then she knows that the jewel is one of the three that weighed more. In either case, after only one weighing, Alley is able to identify a group of only three stones among which is the Rama Nujan.
She then takes two of these three stones and places one on each side of the second scale. If one weighs more than the other, then she knows that this stone is the one containing the jewel. If they both weigh the same, then she knows that the third stone must contain the jewel. Thus, by weighing the stones only twice, Alley is able to find the jewel.

Take partial steps whenever possible. Notice that, instead of trying to identify the jewel immediately, Alley first reduces the pool of choices from nine to three. Thus she first makes the problem easier. “Divide and conquer” is an important and useful technique in both mathematics and life.

**Story 2. Damsel in Distress**

Focusing attention on the corner of the moat suggests using one of the beams to span the corner. Of course, we need to check that the two 19-foot beams are long enough to make the configuration in the picture.

There are at least two ways to verify that this picture is correct. One way is to construct a physical model. The picture shown at left is a physical model scaled down so that 1 foot in the story corresponds to 1 millimeter in the picture. You can now measure and ensure that this configuration is possible.

An alternative method would be to observe that the picture has some right triangles. This observation foreshadows our look at the Pythagorean Theorem. After we examine good old Pythagoras’s theorem (Chapter 4), the following paragraphs will seem soothing and comforting. If for now you find them less so, feel free to glance through them and just move on.

Notice that the corner of the moat forms a 20-foot-by-20-foot square. By the Pythagorean Theorem, the distance from the outer corner of the shore to the inner corner of
the castle island is equal to the square root of $20^2 + 20^2$. Using a calculator, we see that the distance is $28.2842 \ldots$ feet.

Placing the 19-foot beam diagonally across the corner of the moat as far out as it can go creates a triangle that cuts off the corner. If we draw a line from the center of the beam to the outer corner of the moat, we create two identical 45-degree right triangles, as shown. Since the length of half the beam is 9.5 feet, we learn that the center of the beam is also 9.5 feet from the outer corner of the moat.

Since the total diagonal distance from the outer corner of the moat to the corner of the castle island is $28.2842 \ldots$ feet, the distance to the center of the beam is $(28.2842 \ldots \text{ feet} - 9.5 \text{ feet}) = 18.7842 \ldots \text{ feet}$. Since that distance is just less than 19 feet, the other beam will just barely span the distance between the beam and the island. In gratitude for her rescue, the damsel provided the good knight with a romantic lesson in geometry.

**Story 3. The Fountain of Knowledge**

Suppose we fill up the 10-ounce glass with mango juice and slowly pour it into the 6-ounce glass, stopping at the moment the 6-ounce glass is full. Notice that what’s left in the 10-ounce glass is precisely 4 ounces of mango juice. We now empty the 6-ounce glass back into the pool and refill it with the 4 ounces from the other glass. If we now refill the 10-ounce glass from the pool, we can again slowly pour its contents into the 6-ounce glass until the 6-ounce glass is full. Filling it takes exactly 2 more ounces, and now the larger glass contains exactly 8 ounces. Happily, those 8 ounces of mango juice can now be served to Trey (on a tray). If Trey had found a solution, he would have made his first discovery in an area of mathematics known as *number theory.*
There is more than one solution to this puzzle. For example, we could have begun by filling the 6-ounce glass and pouring its entire contents into the 10-ounce glass. See if you can use this starting point to find an alternative solution.

**Story 4. Dropping Trou**

The sequence of diagrams below illustrates a solution to this knotty puzzle. Notice that by bending, contorting, and twisting your pants around, you can produce different configurations. Questions involving bending, contorting, and twisting lead to interesting and surprising discoveries. The notion of bending space is the fundamental notion in an area of mathematics called **topology**.

Often, thinking only in the abstract does not reveal new insights. Make the issue concrete and physical whenever possible.

Many people believe mathematical issues exist outside the realm of our life experience. In truth, many surprising and even counterintuitive mathematical discoveries can be made by freeing ourselves from old, unsubstantiated biases and experimenting with new ways of thinking and seeing.
**Story 5. Dodgeball**

We want to be Player Two. Here is a strategy that will guarantee victory. Player One fills in the first row of six boxes in his table. As Player Two, we look at the first letter and ignore the last five. If his first letter is an X, we write an O; if it’s an O, we write an X. Notice that, no matter what happens later, after this point, we are certain that the row we will create will definitely not be the same as Player One’s first row. The two rows will differ in at least the first box. Player One now writes down his second row of six letters. We examine only the second letter in this new row. If that letter is an X, we write an O; if that letter is an O, we write an X. Now we are sure that no matter what follows, our row will not be the same as Player One’s second row because the rows definitely differ in the second letter. If we repeat this process, we will have created a row of X’s and O’s that is different from the six rows created by Player One.

Creating a row that does not match any of our opponent’s rows has a powerful application in the study of infinity. Although this modest little game has only six steps, the concept behind it has tremendous ramifications, as we shall see in Chapter 3; “Infinity.”

As a final note, we pose the following question: Suppose that we are Player One, and our opponent—who is trying to follow the strategy described above to win—makes a mistake by placing the wrong letter in the first box. Can you now describe a strategy for us, as Player One, to ensure a win? Give this new challenge a try.

**Story 6. A Tight Weave**

The solution is to repeat the process infinitely often. We start with a purple square. At the first stage, a single gold square of size $1/3 \times 1/3$ is placed in the center. At the next stage eight more gold squares of size $1/9 \times 1/9$ are placed in the centers of each of the eight surrounding squares. At the next stage, $8 \times 8$, or 64, more gold squares of size $1/27 \times 1/27$ are placed in the centers of each of the eight squares that surround each of the eight squares that surround the original square. At each stage, we add increasingly many gold squares, each of a smaller size. So the final picture actually has infinitely many gold squares, but each of the eight squares surrounding the central square is an exact replica, though smaller, of the whole picture. This intricate purple and gold carpet is an example of a self-similar object known as a fractal. In Chapter 6, “Fractals and Chaos,” we will examine many such infinitely intricate objects.
What is the area of all the (infinitely many) gold squares? Since all those gold squares lie within the rug that is 1 yard square, we know the area cannot be more than 1. At the first stage, we have one gold square of size $1/3 \times 1/3$, so its area is $1/9$. At the next stage, we add eight more gold squares, each of size $1/9 \times 1/9$, so their areas total $8 \times (1/9)^2$, making the total area of gold squares at stage two equal to $1/9 + 8 \times (1/9)^2 = 0.2098 \ldots$. At the third stage, we add $8^2$ more squares, each of area $(1/9)^3$. Thus, the total area of gold squares at stage three equals $1/9 + 8 \times (1/9)^2 + 8^2 \times (1/9)^3 = 0.2976 \ldots$. Repeating, we begin to see a pattern. The fourth stage, for example, would have a gold area equal to $1/9 + 8 \times (1/9)^2 + 8^2 \times (1/9)^3 + 8^3 \times (1/9)^4 = 0.3757 \ldots$. Thus, the total area of gold squares in the final pattern would be the infinite sum:

$$\frac{1}{9} + 8 \times \left(\frac{1}{9}\right)^2 + 8^2 \times \left(\frac{1}{9}\right)^3 + 8^3 \times \left(\frac{1}{9}\right)^4 + 8^4 \times \left(\frac{1}{9}\right)^5 + 8^5 \times \left(\frac{1}{9}\right)^6 + \ldots .$$

What does it equal? Even though there are infinitely many terms, we know that the whole area must be a number not greater than 1. What number is it?

The gold area at the 5th stage is 0.4450 \ldots ;
- at the 10th stage it is 0.6920 \ldots ;
- at the 15th stage it is 0.8291 \ldots ;
- at the 25th stage it is 0.9474 \ldots ;
- at the 50th stage it is 0.9972 \ldots ;
- at the 100th stage it is 0.999992 \ldots .

From this pattern of numbers, it becomes clear that the gold area becomes increasingly close to 1—and that is a great guess for the area.

Data can help uncover surprising observations and help build intuition and understanding.
A clever way to calculate the total area is to add up all the infinitely many terms. We start by giving a name to the total; let’s call that number \( \text{SUM} \). On the next page you see the infinite sum that \( \text{SUM} \) represents. Directly under that, you see what \((8/9)\text{SUM}\) equals. Notice that multiplying each term of \( \text{SUM} \) by \( 8/9 \) just shifts that term to the right. For example, \((8/9)(1/9) = 8 \times (1/9)^2\).

\[
\text{sum} = \frac{1}{9} + 8 \times \left(\frac{1}{9}\right)^2 + 8^2 \times \left(\frac{1}{9}\right)^3 + 8^3 \times \left(\frac{1}{9}\right)^4 + 8^4 \times \left(\frac{1}{9}\right)^5 + 8^5 \times \left(\frac{1}{9}\right)^6 + \ldots.
\]

\[
\left(\frac{8}{9}\right)\text{sum} = 8 \times \left(\frac{1}{9}\right)^2 + 8^2 \times \left(\frac{1}{9}\right)^3 + 8^3 \times \left(\frac{1}{9}\right)^4 + 8^4 \times \left(\frac{1}{9}\right)^5 + 8^5 \times \left(\frac{1}{9}\right)^6 + \ldots.
\]

Since all the terms of \((8/9)\text{SUM}\) are directly under an identical term of \( \text{SUM} \), it is easy to subtract \((8/9)\text{SUM}\) from \( \text{SUM} \), because all the terms drop out except the first term:

\[
\text{SUM} - \left(\frac{8}{9}\right)\text{SUM} = \frac{1}{9} \quad \text{and so:} \\
\left(\frac{1}{9}\right)\text{SUM} = \frac{1}{9}
\]

Since \((1/9)\text{SUM} = 1/9\), what is \( \text{SUM} \)? It must equal 1! In other words, the area of the gold squares is equal to the area of the entire rug. Thus, even though there are many purple threads remaining in the final pattern, as we begin to see in the illustration on page 25, the purple contributes no area to the rug. Surprise! We will see many more counterintuitive mysteries of infinity in our studies of numbers, fractals, and, of course, infinity itself.

**Story 7. Let’s Make a Deal**

Fortunately, Warren Piece enjoys mathematics as a hobby, so he believes he can solve this conundrum. He thinks carefully, assesses the chances each way, and confidently proclaims (while still jumping up and down, of course), “I switch my guess to Door Number 1, Monty.”

Monty Hall turns and says, “Okay. Let’s see what deal you’ve made. What is behind Door Number 1?” The door swings slowly open, and the
crowd gasps as they see behind Door Number 1 the most beautiful finned chassis that General Motors ever painted pink. Bedlam reigns. “How did you know?” asks Monty Hall over the din. Warren Piece explains.

“When I originally guessed Door Number 3, I had a one-third chance of being right and a two-thirds chance of being wrong. Thus it’s more likely I was wrong than right. When you opened Door Number 2 and revealed no car, I hoped I was wrong originally—which, recall, was more likely than not. If I was wrong originally, then the car must be behind the remaining door, Door Number 1. So I switched, knowing that the probability of my winning after switching was 2 out of 3, whereas the chance of my having picked it correctly the first time was only 1 out of 3.” Monty Hall was compelled to ask, “How did you figure that out?” — to which our hero raisin sagely replied, “By studying The Heart of Mathematics: An invitation to effective thinking.” (Not a bad coast-to-coast TV plug for our book, even coming from a guy named Warren Piece in a raisin suit.)

Some people might think that it doesn’t matter if he switches or not. However, the chances of finding the car are indeed greater by switching. One way to demonstrate this is to list all the possible ways the Cadillac and the mules can be placed behind the doors:

<table>
<thead>
<tr>
<th>Door Number 1</th>
<th>Door Number 2</th>
<th>Door Number 3</th>
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<tr>
<td>Case 1</td>
<td>Cadillac</td>
<td>mule</td>
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<tr>
<td>Case 2</td>
<td>mule</td>
<td>Cadillac</td>
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<tr>
<td>Case 3</td>
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Warren Piece first picked Door Number 3, and the likelihood of his finding the car there was 1 out of 3, that is, one-third, which is not too likely. Next, Monty opened a door showing a mule. Let’s see what happens if Warren were to switch in each of the three possible scenarios. In Case 1, Monty opens Door Number 2. If Warren switches in this case, he wins the Cadillac. In Case 2, Monty opens Door Number 1. If Warren switches in this case, he would win. In Case 3, Monty could open either Door Number 1 or 2. If Warren switches in this case, sadly, he would be the owner of a mule. Therefore, overall, the likelihood of his winning the car by switching is 2 out of 3, or two-thirds, which is twice as likely as the one-third chance of selecting the car if he sticks to his original guess. This brief encounter with probability illustrates that counterintuitive outcomes can occur during attempts to measure the unknown.
Story 8. Rolling Around in Vegas

The die with four 4’s and two blanks beats the die with all 3’s two-thirds of the time. There are four 4’s on the 4-0 die so it comes up 4 two-thirds of the time it is rolled. Whenever it comes up 4 it beats the 3-die, which always comes up 3. So the die with four 4’s and two 0’s will win two-thirds of the time.

The die with three 5’s and three 1’s will beat the die with four 4’s and two blanks, because one-third of the time it is rolled, the 4-0 die will come up with a 0, in which case it loses, no matter what the 5-1 die comes up. In two-thirds of the cases where the 4-0 die comes up with a 4, half those times, or one-third of the time that the 4-0 die is rolled, the 5-1 die will come out with a 5, in which case the 5-1 die wins anyway. So altogether, two-thirds of the time the 5-1 die wins over the 4-0 die. We can also see this fact by making a chart of all 36 outcomes of rolling the two dice against each other and noticing that in 24 of those outcomes the 5-1 die wins. Those 24 winning squares are colored in gold in the chart below.

The die with the two 6’s and four 2’s will beat the die with three 5’s and three 1’s, because one-third of the time it is rolled, the 6-2 die will come up with a 6, in which case it wins no matter what the 5-1 die comes up. In two-thirds of the cases where the 6-2 die comes up with a 2, half those times, or one-third of the time the 6-2 die is rolled, the 5-1 die will come out with a 1, in which case the 6-2 die wins anyway. So altogether, two-thirds of the time the 6-2 die wins over the 5-1 die. Again, we can also see this fact by making a chart of all 36 outcomes of rolling the two dice against each other and noticing that in 24 of those outcomes the 6-2 die wins. Those 24 winning squares are colored in gold in the chart on the next page.
Finally, the die with all 3’s can beat the die with two 6’s and four 2’s two-thirds of the time. To see this, just note that a 2 will appear two-thirds of the time that the 6-2 die is rolled, and each time a 2 appears, the 3-die will win.

So in the following ordering, each die beat the one next to it in the clockwise direction two-thirds of the time:

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**Story 9. Watsamattawith U?**

Suppose there are 300 students in each of the graduating classes. In last year’s class, 150 were male and 150 were female. Suppose that the average GPA of last year’s men was 2.0 and the average GPA of last year’s women was 3.5. Then the average GPA for last year’s graduating class was 2.75. One way to arrive at that answer is to replace all the men’s GPAs with the male average and all the women’s GPAs with the female average and then average all 300 GPAs as follows:

\[
\frac{(2.0 \times 150 + 3.5 \times 150)}{300} = \frac{825}{300} = 2.75.
\]

Now assume that this year’s graduating class is comprised of 200 men and only 100 women. If the average GPA of the male students rose to 2.1 (higher than last year’s average) and the average GPA of the female
students became 3.6 (again higher than last year’s average), then the GPA for this year’s class can be found by computing: \( \frac{2.1 \times 200 + 3.6 \times 100}{300} = \frac{420 + 360}{300} = \frac{780}{300} = 2.6 \)!

Amazing . . . until we realized we had the ability to change the proportion of men to women. By having a higher proportion of the poorer male students, even though the GPA of the males increased from 2.0 to 2.1, since there were more males this year, they dragged down the GPA of the student body. The number of male students compared to female students was a quantity that could be changed, but we might not have thought about that possibility. Hidden features of a statistical situation like this are sometimes called “lurking variables.”

**Story 10. Dot of Fortune**

The math fan sees a red dot on the forehead of each of the other two players. She knows she has either a white dot or a red dot on her own forehead. Let’s see what happens if we suppose her dot is white.

What would her two companions at the table see? Each would see one red dot and one white dot, and each would see two arms raised. Each would be thinking, “Do I have a red dot or a white dot on my forehead? If I have a white dot, then the red-dotted person would not have her hand up. Therefore, I must have a red dot.” After making this easy deduction, this person would hit the buzzer.

But what did these two people actually do? Or, more to the point, what did they not do? They did not hit their buzzers! If either of them had seen a white dot and a red dot and two raised hands, he or she would have been able to deduce that his or her own dot was red. Since neither person buzzed right away, neither must have seen a white dot on the math fan’s forehead. Therefore, the math fan waited just long enough to know that the other two players could not deduce their own dot colors, and then she buzzed, confident that her dot was red.

A final question of the story is, Why did the other students not know? The answer to that question is, of course, because they had not read *The Heart of Mathematics.*
1.4 FROM PLAY TO POWER

Discovering Strategies of Thought for Life

Imagination is more important than knowledge.
ALBERT EINSTEIN

Grandmaster Maurice Ashley, teaching chess strategies to inner-city children in Harlem.

Our stories illustrate strategies of thinking. Even in such a lighthearted setting, certain techniques of thought emerge as powerful means to illuminate the unknown—techniques applicable to any situation we may face in life. We’ll encounter more “life lessons” elsewhere in *The Heart of Mathematics*; on the next page we’ve summarized a few. Although some may seem obvious or trivial, don’t take them lightly—they can be surprisingly useful for analyzing and enjoying life’s adventure.
Lessons for Life

1. Just do it.
2. Make mistakes and fail, but never give up.
4. Explore the consequences of new ideas.
5. Seek the essential.
6. Understand the issue.
7. Understand simple things deeply.
8. Break a difficult problem into easier ones.
9. Examine issues from several points of view.
10. Look for patterns and similarities.

Mindscapes Invitations to Further Thought

We now provide some additional stories for further amusement and enlightenment. We call them “Mindscapes” because they are vistas for the mind that encourage you to expand your way of thinking.

For each of the following situations, contemplate, analyze, and resolve the puzzle. Also, guess which branch of mathematics each situation represents: Logic, Number Theory, Infinity, Geometry, Topology, Chaos, or Probability. Of course, we haven’t discussed any of these areas in depth yet, but just take a guess—being wrong is fine.

Finally, we invite you to provide an aesthetic critique of each question and your solutions. In other words, did you find either the question or your solution interesting? Which questions were the most challenging? Do you like one of your solutions better than the others? At the end of this section we provide some hints for some of the questions. Use them sparingly.
1. **Late-night cash.** Suppose that David Letterman and Paul Shaffer have the same amount of money in their pockets. How much must Dave give to Paul so that Paul would have $10 more than Dave?

2. **Politicians on parade.** There were 100 politicians at a certain convention. Each politician was either crooked or honest. We are given the following two facts:
   a. At least one of the politicians was honest.
   b. Given any two of the politicians, at least one of the two was crooked.

   Can it be determined from these facts how many of the politicians were honest and how many were crooked? If so, how many? If not, why not?

3. **The profit.** A dealer bought an item for $7, sold it for $8, bought it back for $9, and sold it for $10. How much profit did she make?

4. **The truth about . . .** Fifty-six biscuits are to be fed to 10 pets; each pet is either a cat or a dog. Each dog is to get six biscuits, and each cat is to get five. How many dogs are there? (Try to find a solution without performing any algebra.)

5. **It’s in the box.** There are two boxes: one marked A and one marked B. Each box contains either $1 million or a deadly snake that will kill you instantly. You must open one box. On box A there is a sign that reads: “At least one of these boxes contains $1 million.” On box B there is a sign that reads: “A deadly snake that will kill you instantly is in box A.” You are told that either both signs are true or both are false. Which box do you open? Be careful! The wrong answer is fatal!

6. **Lights out.** Two rooms are connected by a hallway that has a bend in it so that it is impossible to see one room while standing in the other. One of the rooms has three light switches. You are told that exactly one of the switches turns on a light in the other room, and the other two are not connected to any lights. What is the fewest number of times you would have to walk to the other room to figure out which switch turns on the light? And the follow-up question is: Why is the answer to the preceding question “one”? (Look out: This question uses properties of real lights as well as logic.)

“Contrariwise,” continued Tweedle-dee, “if it was so, it might be; and if it were so, it would be; but as it isn’t, ain’t. That’s logic.”
LEWIS CARROLL
7. **Out of sight but not out of mind.** The infamous band Slippery Even When Dry ended their concert and checked into the Fuzzy Fig Motel. The guys in the band (Spike, Slip, and Milly) decided to share a room. They were told by Chip, the night clerk who was taking a home-study course on animal husbandry, that the room cost $25 for the night.

Milly, who took care of the finances, collected $10 from each band member and gave Chip $30. Chip handed Milly the change, $5 in singles. Milly, knowing how bad Slip and Spike were at arithmetic, pocketed two of the dollars, turned to the others, and said, “Well guys, we got $3 change, so we each get a buck back.” He then gave each of the other two members a dollar and pocketed the last one for himself.

Once the band members left the office, Chip, who witnessed this little piece of deception, suddenly realized that something strange had just happened. Each of the three band members first put in $10 so there was a total of $30 at the start. Then Milly gave each guy and himself $1 back. That means that each person put in only $9, which is a total of $27 ($9 from each of the three). But Milly had skimmed off $2, so that gives a total of $29. But there was $30 to start with. Chip wondered what happened to that extra dollar and who had it. Can you please resolve and explain the issue to Chip?

8. **Comedy Central.** A good comedian armed with strong material can really kill an audience. Bad comedians, on the other hand, are caught and brought to Comedy Central Correctional (CCC), where they do their timing. But moving C- and D-list comedians to lockdown is not as easy as it may appear. One day, in the outback of Los Angeles, three comedy correctional officers were escorting three bad comedians, who had just bombed at the Laff Stop, to CCC. Suddenly, along their march to CCC, they came upon the banks of the mighty Los Angeles River and needed to cross its turbulent and deep waters. None of the six could swim, but all could row. Fortunately, on the river’s shore was a small rowboat available for use.

Since the boat was small and the officers and comedians were all on the portly side, it was clear that only two persons could cross at one time. The problem was: While each comedian was not particularly great, if ever there were a moment when the comedians outnumbered the officers, then the comedians—together—were funny enough to kill their audience (*i.e.*, the officers). Given this reality, the officers decided that being prudent was better than being dead, so they agreed that at no time would they allow any group of officers to be outnumbered by comedians during the crossing. For their part, the comedians did not fear being outnumbered by the officers because they realized that an excess of officers would result only in more discussion among the officers, thus relieving the comedians of the burden of being “on.”
How do the officers and comedians all cross the river using only the one boat yet at no time letting the comedians outnumber the officers on either side of the river?

9. **Whom do you trust?** Congresswoman Smith opened the *Post* and saw that a bean-counting scandal had been leaked to the press. Outraged, Smith immediately called an emergency meeting with the five other members of the Special Congressional Scandal Committee, the busiest committee on Capitol Hill.

Once they were all assembled in Smith’s office, Smith declared, “As incredible as it sounds, I know that three of you always tell the truth. So now I’m asking all of you, Who spilled the beans to the press?”

Congressman Schlock spoke up, “It was either Wind or Pocket.”

Congressman Wind, outraged, shouted, “Neither Slie nor I leaked the scandal.”

Congressman Pocket then chimed in, “Well, both of you are lying!”

This provoked Congressman Greede to say, “Actually, I know that one of them is lying and the other is telling the truth.”

Finally, Congressman Slie, with steadfast eyes, stated, “No, Greede, that is not true.”

Assuming that Congresswoman Smith’s first declaration is true, can you determine who spilled the beans?

10. **A commuter fly.** A passenger train left Austin, Texas, at 12:00 p.m. bound for Dallas, exactly 210 miles away; it traveled at a steady 50 miles per hour. At the same instant, a freight train left Dallas headed for Austin on the same track, traveling at 20 miles per hour. At this same high noon, a fly leaped from the nose of the passenger train and flew along the track at 100 miles per hour. When the fly touched the nose of the oncoming freight train, she turned and flew back along the track at 100 miles per hour toward the passenger train. When she reached the nose of the passenger train, she instantly turned and flew back toward the freight train. She continued turning and flying until, you guessed it, she was squashed as the trains collided head on.

How far had the fly flown before her untimely demise?

11. **A fair fare.** Three strangers, Bob, Mary, and Ivan, meet at a taxi stand and decide to share a cab to cut down the cost. Each has a different destination, but all the destinations are on the highway leading from the airport, so no circuitous driving is required. Bob’s destination is 10 miles away, Mary’s is 20 miles, and Ivan’s is 30 miles. The taxi costs $1.50 per mile including the tip, regardless of the number of passengers. How much should each person pay? (*Caution: There is more than one way of looking at this situation.*)
12. **Getting a pole on a bus.** For his 13th birthday, Adam was allowed to travel down to Sarah’s Sporting Goods store to purchase a brand new fishing pole. With great excitement and anticipation, Adam boarded the bus on his own and arrived at Sarah’s store. Although the collection of fishing poles was tremendous, there was only one pole for Adam and he bought it: a 5-foot, one-piece fiberglass “Trout Troller 570” fishing pole.

When Adam’s return bus arrived, the driver reported that Adam could not board the bus with the fishing pole. Objects longer than 4 feet were not allowed on the bus. Adam remained at the bus stop holding his beautiful 5-foot Trout Troller. Sarah, who had observed the whole ordeal, rushed out and said, “We’ll get your fishing pole on the bus!” Sure enough, when the same bus and the same driver returned, Adam boarded the bus with his fishing pole, and the driver welcomed him aboard with a smile. How was Sarah able to have Adam board the bus with his 5-foot fishing pole without breaking or bending the bus-line rules or the pole?

13. **Tea time.** Carmilla Snobnosey lifted the delicate Spode teapot and poured exactly 3 ounces of the aromatic brew into the flowered, shell china teacup. She placed the cream pitcher, also containing exactly 3 ounces, on the Revere silver tray and carried the offering to Podmarsh Hogslopper.

“Would you like some tea and cream, Mr. Hogslopper?” she asked.

“Yup. Thanks. Ow doggie, sure looks hot. I’d better cool it down with this here milk,” he responded politely and carefully poured exactly 1 ounce of cream into his steaming tea and stirred. “That oughta do it,” he said when the steam stopped rising from the tea. “Here, I’ll just give you back that there cream.” Whereupon he carefully spooned exactly 1 ounce from his teacup back into the creamer. Podmarsh blushed as he looked at a tea leaf or two floating in the cream and realized his faux pas. Caught at an awkward pass, he decided to smooth things over with an intriguing puzzle.

“Ya know, Mrs. Snobnosey, I wonder if the tea is more diluted than the cream, or if the cream is more diluted than the tea?”

Resolve the dilution problem.

14. **A shaky story.** Stacy and Sam Smyth were known for throwing a heck of a good party. At one of their wild gatherings, five couples were present (this included the Smyths, of course). The attendees were cordial, and some even shook hands with other guests.

Although we have no idea who shook hands with whom, we do know that no one shook hands with themselves and no one shook hands with his or her own spouse. Given these facts, a guest might not
shake anyone’s hand or might shake as many as eight other people’s hands. At midnight, Sam Smyth gathered the crowd and asked the nine other people how many hands each of them had shaken.

Much to Sam’s amazement, each person gave a different answer. That is, someone didn’t shake any hands, someone else shook one hand, someone else shook two hands, someone else shook three hands, and so forth, down to the last person, who shook eight hands. Given this outcome, determine the exact number of hands that Stacy Smyth shook.

15. Murray’s brother. On another archeological dig, Alley discovered another ancient oil lamp. Again she rubbed the lamp, and a different genie named Curray appeared. After Alley explained her run-in with Murray, Curray responded, “Well, since you know my brother Murray, it’s like we’re almost family. I’m going to give you four wishes instead of three. What do you say?” Since things had worked out so well the last time, she said, “I already found the Rama Nujan, so now I’d like to find the Dormant Diamond.” “You got it,” replied Curray. And instantly 12 identical-looking stones appeared. She then used her last three wishes to acquire three balance scales. Each scale was clearly labeled, “One Use Only.” Alley looked at the stones and was unable to differentiate any one from the others. Curray explained, “The diamond is embedded in one of the stones. Eleven of the stones weigh the same, but the stone containing the jewel weighs either slightly more or slightly less than the others. I am not telling you which—you must find the right stone and tell me whether it is heavier or lighter.”

Alley could use each of the three balance scales exactly once. She was able to select the stone containing the Dormant Diamond from among the 12 identical-looking stones and determine whether it was heavier or lighter than each of the 11 other stones. This puzzle is a challenge. Try to figure out how Alley might have accomplished this feat.

Further Challenges

16. Cutting (chess) boards. Suppose we are given a standard $8 \times 8$ checkerboard and an immense supply of dominoes. Each domino can cover exactly two adjacent squares on the checkerboard (first checkerboard on p. 38). As a warm-up, verify that the checkerboard can be covered completely by dominoes where each domino covers exactly two squares and the dominoes do not overlap one another. Assume next that two squares of the checkerboard have been cut off as shown (second checkerboard on p. 38). Your challenge now is to determine if you can cover this cut checkerboard
with nonoverlapping dominoes so that again, each domino covers exactly two squares. Finally, your last challenge is to consider the same question for the truncated checkerboard (last checkerboard). Does your answer change? Justify your answers.

17. Siegfried & You. Consider the following mathematical illusion. A regular deck of 52 playing cards is shuffled several times by an audience member until everyone agrees that the cards are completely shuffled. Then, without looking at the cards themselves, the magician divides the deck into two equal piles of 26 cards. The magician taps both piles of face-down cards three times. Then, one by one the cards of both piles are revealed. Magically, the magician was able to have the cards arrange themselves so that the number of cards showing black suits in the first pile is identical to the number of cards showing red suits in the second pile. Your challenge is to figure out the secret to this illusion and then perform it for your friends.

18. Penny for your thoughts. Some number of pennies are spread out on a table. They lie either heads up or tails up (see the figure on the next page). Unfortunately, you are blindfolded and thus both the coins and the table upon which they sit are hidden from view. You can feel your way across the table and thus can count the total number of pennies on the table’s surface, but you cannot determine if any individual penny rests heads up or down (perhaps you’re wearing gloves). You are informed of one fact (beyond the total number of pennies on the table): Someone tells you the number of pennies that are lying heads up. While remaining blindfolded, you may now rearrange the coins, turn any of them over, and move them in any way you wish as long as the final configuration has all the pennies resting (heads or tails up) on the table. Your challenge is to arrange the pennies into two collections and then turn over whatever pennies you wish so that both collections have the same number of heads-up pennies.
19. **When will the world end?** The Towers of Hanoi is a puzzle consisting of three pegs and a collection of punctured disks of different diameters that can be placed around any of the pegs. The puzzle begins with all the disks on a single peg in descending order of diameter, with the largest disk on the bottom (top figure, p. 40). The object is to transfer all the disks to another peg so that they end up residing on this new peg in the original descending order given the following two rules: Every move consists of removing the top disk on one peg and placing it on top of the pile on another peg; and at no time can a larger disk be placed on top of a smaller disk (bottom figure, p. 40). Describe a solution to the puzzle if there are four disks, then again if there are five disks, and again if there are six disks. Can you discover a pattern to the minimum number of moves required to solve the puzzle, given how many disks there are?
There is a legend that certain monks had a magnificent edition of this puzzle consisting of 64 gold disks and three diamond pins. They were able to move one disk per second. The legend is that the world would end once the monks completed their mission. Use the pattern you found to predict when the world will end—a useful piece of information as you plan your future.

20. **The fork in the road.** You are vacationing on a mythical island resort in which it never rains and they get all the cable stations you know and love. One day you decide to actually turn off the TV and go outside for a hike. Soon you find yourself lost in a forest. You yearn for the main ingredient of the Food Network and must get back to your resort pronto. You finally arrive at a fork in the road (which again reminds you of food). You know that one path will take you back safely to your villa and the other one will lead you into a den of tigers where you will play the role of the main course. You have no idea which road to take. The good news is that you see a native by the fork in the road who knows which road leads to the resort; the bad news is that the natives come from one of two tribes: the Liars or the Truth-Tellers. The Liars always lie, while the Truth-Tellers always tell the truth. The other bad news is that there is no
way to tell which tribe this native belongs to. No matter which tribe, however, the natives are sick of all the silly tourists (like you) and all their TV talk. They can only stand one question—that is, you can only ask one question in order to find your way back. What question do you ask the native?

**Hints and Problem-Solving Techniques**

Here are some suggestions for tackling these puzzles. Hints and solutions to selected Mindscapes in later chapters appear at the end of the book.

1. **Late-night cash.** Try it! After you act it out, explain what happened.
2. **Politicians on parade.** What if more than one politician is honest? Read fact (b) carefully.

3. **The profit.** Different people will get different answers, and each person will argue that his or hers is correct. Act out the transactions and see what happens. After you try this, go back and figure out why other answers are incorrect.

4. **The truth about . . .** What if all the animals were cats? How many extra biscuits would you have? Consider turning some of those cats into dogs. This transformation leads to an algebra-free solution.

5. **It’s in the box.** Consider the two possibilities carefully. You don’t want to slip up on this one.
6. **Lights out.** Suppose you turn on a switch, wait a half hour, and then turn the switch off. If you were then to walk into the other room, could you tell if the light had been on for half an hour? Ponder this question, and use it to resolve the original puzzle.

   *Don't overlook or dismiss facts that seem insignificant or irrelevant.*

7. **Out of sight but not out of mind.** Don’t be fooled by all the numbers. Force yourself to figure out what was paid out and what was given back.

   *Don’t believe unsubstantiated claims, even if they sound scientific. Until you understand the issue for yourself, be skeptical!*

If that doesn’t help, get 30 $1 bills and act out the entire episode. Once you discover the truth, go back and find out where the problem is in the story.

   *Experimentation is a powerful means for discovering patterns and developing insights.*

See how many different ways you can devise to understand and explain what actually happened.

   *Once you find an argument that resolves an issue, it is a great challenge to find a different argument. However, in attempting to find other arguments, we often gain further insight into and understanding of the situation. Also, the first argument we come up with may not be the best one.*

8. **Comedy Central.** Professor Starbird shares his grandmother’s solution.

   When my grandmother was 92, I gave this challenge to her along with three nickels to represent the comedians and three Life Savers candies to represent the officers. We set up a line on her table to represent the river so that she could slide the comedians and the officers (the nickels...
and the Life Savers) back and forth singly or in pairs, thereby solving the puzzle. When I arrived the next day for my visit, she was delighted to tell me that she had solved the puzzle.

“How did you do it?” I asked.

She replied triumphantly, “I ate the officers.”

We give her half credit. A useful aspect of her method was to model the puzzle using a concrete representation. Making a written table with two columns would also be a good way to represent the setting. One column would be one bank of the river, and the other column would be the other bank. Each row would represent the situation after a crossing. So the first row would have three C’s, three O’s, and a B for the boat in the left-hand column and nothing in the right. The next row might have two C’s and two O’s in the left column and one C, one O, and the B in the right column. Going from row to row must be possible by moving one or two C’s or O’s along with the B to the other column.

9. **Whom do you trust?** To find the person who leaked the story, you must determine who is telling the truth. Ask yourself whether you can determine the truthfulness or deceit of any one person.

If Pocket is telling the truth, then Schlock and Wind are liars, and the remaining three—Pocket, Greede, and Slie—are telling the truth. Could those three all be telling the truth? If not, then you know for certain that Pocket is lying.

Since Slie contradicts Greede, you know that one of them is lying. Which one?

**A rock of certainty can be the foundation of a tower of truth.**
10. **A commuter fly.** On close inspection, notice that the fly changes directions an infinite number of times during her travels. It is possible to compute how far the fly has flown before she encounters the freight train for the first time. Once you know this, it’s possible to compute the distance she travels before encountering the passenger train on the return trip. You could compute those distances and find a pattern and then solve the problem by adding up the infinite list of distances. However, there is a much easier way to solve this puzzle.

How much time will pass before the trains collide? How far will the fly fly in that length of time? Case closed.

This story is not complete without our telling an anecdote about the famous mathematician John von Neumann. Von Neumann was notorious for being extremely fast and accurate at calculating numbers in his head—oddly enough, not a skill that all mathematicians possess. One day he was walking with a friend who asked him the question of the fly between the trains. Instantly, von Neumann stated the answer. The questioner said, “Oh, you saw the trick.” To which von Neumann replied, “Yes, it was an easy infinite series.”

If you are not von Neumann, the fly-between-the-trains story provides a good life lesson.

Go out of your way to think about different ways to view a problem. In this case, if you know how long the fly flies, you can compute the distance the fly travels. You have now reduced the original problem to a different, though related, problem. In this case, the different problem is much simpler to solve than the original one.

11. **A fair fare.** This question does not have one definitive answer. However, a look at a related problem may persuade you that one possibility is best. What if, instead of staying in one taxi the whole time, the three travelers traveled the first 10 miles together and then all got out and paid the first cabby. The first traveler then left, and the remaining two got another cab, rode 10 miles, and again got out and paid the second cabby. Then the last traveler took a cab alone for the
remaining 10 miles. This rephrasing of the original problem makes the division of payment seem more obvious.

12. **Getting a pole on a bus.** It seems impossible to get the 5-foot pole on the bus, given that the largest length of an item allowed on the bus is 4 feet. Sarah gave Adam a large box to put the pole in. Now give the dimensions of the box and explain why it does the trick.

13. **Tea time.** This question contains much unnecessary and distracting information. A close look at the story reveals that the description of the dinnerware and the names of the people are extraneous details. But what may not be quite so obvious is that the number of ounces in the teacup and the creamer and the amounts poured and spooned are also irrelevant.

Don’t be distracted by this extraneous information. Suppose the problem did not contain those facts at all and instead was stated as follows:

A creamer and a teacup each have exactly the same amount of cream and tea, respectively. An undisclosed amount of mixing of the cream and tea goes on, but after the mixing, each of the two containers still contains the same amount of liquid as the other. Is the tea more diluted than the cream, or is the cream more diluted than the tea?

Having less information might force you to look at the situation differently and, consequently, to understand and solve it.

14. **A shaky story.** Exactly one person at the party said to Sam that he or she shook eight hands. Note the obvious fact that each person with whom that person shook hands must have shaken hands with at least one person. Now determine how many hands that person’s spouse shook. See if this approach leads to any insights. If it still does not, consider an easier problem: Suppose that there were just three couples, or even two couples. Search for a pattern.
15. Murray’s brother. This genie has posed a difficult challenge. It can be fun to work on, but do not work on it too long if you get frustrated. In this puzzle, we must squeeze every ounce (or even gram) of information from every weighing.

If you have a hard problem, first work on a simpler, related problem to develop insight.

Don’t ignore information.

Each weighing must be designed to give us maximum information. After a weighing, we learn many things. Let’s begin by putting four stones on each side of a scale and recording what we observe. If the scale balances, we know that all eight stones weigh the same, and the diamond is not among those eight. So the mystery stone is among the remaining four, but we still do not know whether it is heavier or lighter than the others. Can you now find the Dormant Diamond and determine whether it is heavy or light?

Suppose that the four-against-four weighing does not balance. This imbalance gives us much information. We know that the unweighed four stones all weigh the same. We know that each of the four stones on the light side of the scale are potentially light, but none of them is potentially heavier than the 11 other stones. We know similar things about the four stones on the other side of the scale. We will have to keep track of the stones and consider putting potentially light stones with potentially heavy ones to help sort things out. For example, suppose we weigh a potentially light stone with a potentially heavy stone on one side of the scale and two stones that are known to be normal on the other side. Then, depending on which way the scale tips, we can conclude which of the two stones is the Dormant Diamond.

You might think about the last step to help you find an intermediate solution. That is, you might specify what collections of stones and knowledge would allow you to find the diamond in one more weighing. For example, suppose you figure out that the diamond is among three stones that are potentially heavier than the others. Could you find the diamond in one more weighing? Or suppose you had narrowed the field to three stones, one potentially heavier than normal and two potentially lighter than normal. Could you find the diamond in one more weighing? This technique of working backward is often useful.
This balance-scale conundrum is tricky and difficult. Everyone, including experienced mathematicians, would have to think hard to solve it. It can be fun to work on if you enjoy this type of puzzle. Play with it; think carefully about what you know; carefully keep track of all the information you gather. But if you’re not enjoying yourself, then just move on.

16. **Cutting (chess) boards.** In the first truncated chessboard, only the top row has been changed. Can you cover this new row with dominoes? Doesn’t seem too hard, right? The second truncated board is a different story. How would your covering method change? If you’re having trouble finding a successful covering, can you think of a clear explanation as to why such a covering is impossible? Look closely at what lies underneath a domino that covers two squares. Then look at the deleted squares of the second truncated board.

17. **Sigfried & You.** It’s very important to recall that a deck of cards has 26 black cards and 26 red cards. Consider a particular example of this trick. Suppose it happens that the first pile created by the magician has 10 black cards and 16 red cards. What’s in the second pile? Try another example. What happens in general?

18. **Penny for your thoughts.** First try this challenge with two pennies having one head showing. (It’s OK to have your eyes open while you experiment, but remember that you want a method that works with your eyes closed.) How can you get two collections? Once you’ve created your two collections, can you flip one or two coins so that both collections have the same number of heads? (Remember that zero is a number!)

   Now think about a slightly bigger case: three pennies with one head showing. What sizes will your two collections be? Consider the cases for which collection has the single head. In each case, can you do some flipping to equalize the heads?
Do you have an idea that applies easily to larger examples? Think about the previous challenge: The deck had 26 black cards, so in any pile of 26 cards, each “missing” black card was replaced by a red card, and the missing black cards were all in the other pile. This is what made the trick work.

Suppose you had 10 pennies with six heads showing. Think about what an arbitrary collection of six coins would look like. If it doesn’t contain all the heads, where are the missing heads? How many pennies show tails in your collection of six? Now remember that you are allowed to flip as many coins as you like!

Try an example and look for ways to relate to previous experiences. Consider extreme cases, such as the smallest one.

19. **When will the world end?** Before trying the puzzle with four or five disks, try it with two and then three. With only two disks the puzzle is easy, taking only three moves. As you approach the puzzle with three disks, think about what you have to do before you can move the bottom disk. After you move the bottom disk, what do you have to do to finish? For the puzzle with four disks, what do you have to accomplish before you can move the bottom disk? What about after you move the bottom disk? Do you see the pattern? Fill in the table below and see if you can guess a formula for the number of moves required for a puzzle with $n$ disks. Use your formula to help predict when the world will end.

<table>
<thead>
<tr>
<th>Number of Disks</th>
<th>Number of Moves</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>64</td>
<td></td>
</tr>
</tbody>
</table>

Look at simpler cases and build up; uncover patterns as you go.
20. The fork in the road. Because the goal is to choose left or right, suppose any question you ask will have one of those two words as an answer. The challenge is to create a single question with two properties: Both Liars and Truth-Tellers will give the same answer AND you know what the answer means. That is, you know whether the answer is telling you the road to take or the road not to take.

For now, let’s suppose that the left road returns to the villa and illustrate the situation with a simple table. When we ask the most straightforward question, we get two different answers, so we would not get definitive information. We want to modify our question so that only one of the responses changes. Think about how a native from one tribe would read the answer given by a member of the other tribe. Any ideas?

<table>
<thead>
<tr>
<th>“Which road leads to the villa?”</th>
<th>Better Question:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Answer from Liar</td>
<td>RIGHT</td>
</tr>
<tr>
<td>Answer from Truth-Teller</td>
<td>LEFT</td>
</tr>
</tbody>
</table>

Consider different perspectives within the challenge at hand.