Math 324: Assignment 6

Instructions. Exercises 1, 2, 3 are to be written and turned in on Monday, October 25. Exercises 2(a), 4, 5, 6 and 7 are for in-class presentation.

Exercise 1. Consider the matrix

\[
A = \begin{bmatrix}
1 & 0 & -2 & 1 & 5 \\
3 & 1 & -5 & 0 & 8 \\
1 & 2 & 0 & -5 & -9
\end{bmatrix}
\]

Let the row-space of \(A\) be the subspace of \(\mathbb{R}^5\) generated by the rows of \(A\). Let the column-space of \(A\) be the subspace of \(\mathbb{R}^3\) generated by the columns of \(A\).

(a) Find a basis for the row-space of \(A\).

(b) Express every row in \(A\) as a linear combination of basis vectors found in (a).

(c) Find a basis for the column space of \(A\).

(d) Express every column in \(A\) a linear combination of the basis vectors found in (c).

(e) What are the dimensions of the row-space and column space? Are they equal to each other? Will they always be equal to each other, irrespective of the matrix?

(f) Find the null-space of \(L_A\) and find a basis for the null-space of \(L_A\).

(g) For an \(m \times n\) matrix, the dimension theorem states that

\[n = \dim(N(A)) + \text{rank}(A)\]

Verify that this is true for the matrix above.

Exercise 2. (a) Use matrix multiplication to show that any solution to \(Ax = b\) can be written as \(x = x_p + x_h\) where \(Ax_p = b\) and \(Ax_h = 0\).

(b) Verify this with the system of equations \(Ax = b\) where

\[
A = \begin{bmatrix}
0 & 3 & -6 & -4 & -3 \\
-1 & 3 & -10 & -4 & -4 \\
2 & -6 & 20 & 2 & 8
\end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix}
-5 \\
-2 \\
-8
\end{bmatrix}
\]


Exercise 4. Suppose \(T : V \to W\) is a linear transformation, and \(\{v_1, v_2, \ldots, v_n\}\) is a basis for \(V\). Show that \(\{T(v_1), T(v_2), \ldots, T(v_n)\}\) generates \(R(T)\), the range of \(T\).
Exercise 5. Suppose $A$ is an $m$ by $n$ real matrix. Prove that the columns of $A$ generate the space $R(L_A)$ (i.e. the image of $L_A$).

Exercise 6. Show that $Ax = b$ has at least one solution if and only if $b$ is a linear combination of the columns of $A$.

Exercise 7. Let $A$ be an $m$ by $n$ matrix. How can you tell if $Ax = b$ has infinitely many solutions? (Express your answer in terms of $n$, and the ranks of $A$ and $(A|b)$).