1. p. 322 #2bc.

**Answer.** (b) \( W \) is not a \( T \)-invariant subspace of \( V \) because, for example \( x^2 + 1 \in W \) but \( T(x^2 + 1) = x^3 + x \notin W \).

(c) \( W \) is a \( T \)-invariant subspace of \( V \). Let \( w \in W \) be arbitrary, then \( w = (t, t, t) \) for some \( t \in \mathbb{R} \). Then \( T(w) = (3t, 3t, 3t) = (s, s, s) \) where \( s = 3t \in \mathbb{R} \). Thus \( T(w) \in W \) and so \( W \) is \( T \)-invariant.

2. Let \( V \) be a vector space and let \( T \) be a linear operator on \( V \).

(i) Prove that \( N(T) \) is a \( T \) invariant subspace.

(ii) Prove that \( R(T) \) is a \( T \) invariant subspace.

(iii) Prove that \( E_\lambda \) where \( \lambda \) is an eigenvalue of \( T \) is a \( T \)-invariant subspace.

**Proof.** (i) Let \( v \in N(T) \). Then \( T(v) = 0 \) and \( 0 \in N(T) \). Therefore, \( T(v) \in N(T) \) for any \( v \in N(T) \). Thus \( N(T) \) is \( T \)-invariant.

(ii) Let \( v \in R(T) \). Then \( v \in V \) and so \( T(v) \in R(T) \). Thus \( R(T) \) is a \( T \)-invariant subspace of \( V \).

(iii) Let \( v \in E_\lambda \). Then \( T(v) = \lambda v \in E_\lambda \) because \( E_\lambda \) is closed under scalar multiplication. Thus \( E_\lambda \) is a \( T \)-invariant subspace.

3. p. 322 #6(a).

**Answer.** Observe that \( T(1, 0, 0, 0) = (1, 0, 1, 1) \) and \( T(1, 0, 1, 1) = (1, -1, 2, 2) \) and \( T(1, -1, 2, 2) = (0, -3, 3, 3) = -3(1, 0, 1, 1) + 3(1, -1, 2, 2) \). Thus an ordered basis for the \( T \)-cyclic subspace is \( \{(1, 0, 0, 0), (1, 0, 1, 1), (1, -1, 2, 2)\} \).

4. Let \( A \) be an \( n \) by \( n \) matrix with characteristic polynomial

\[
f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0
\]

(a) Claim: \( A \) is invertible if and only if \( a_0 \neq 0 \).

**Proof.** First, \( a_0 \neq 0 \) if and only if 0 is not a zero of \( f(t) \) if and only if 0 is not an eigenvalue of \( A \) if and only if \( |A - 0I| \neq 0 \) if and only if \( |A| \neq 0 \) if and only if \( A \) is invertible.

(b) Suppose \( A \) is invertible. Then

\[
A^{-1} = (-1/a_0) [(-1)^n A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_1 I_n].
\]
Proof. By part (a), $a_0 \neq 0$ when $A$ is invertible. According to the Cayley-Hamilton theorem $f(A) = 0_n$ where $0_n$ is the $n$ by $n$ 0 matrix. Therefore,

$$(-1)^n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I_n = O_n$$

Thus

$$\frac{-1}{a_0} [(-1)^n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A] = I_n$$

Factoring $A$ out of this, we get

$$\frac{-1}{a_0} [(-1)^n A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_2 A + a_1 I_n] A = I_n.$$ 

Now $AC = I_n$ where $C = (-1/a_0) [(-1)^n A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_1 I_n]$ and hence $A^{-1} = C$ which provides the desired formula.

(c) The characteristic polynomial for $A$ is $f(t) = (1-t)(2-t)(-1-t) = (-1)^3 t^3 + 2t^2 + t - 2$. Thus, by (a),

$$A^{-1} = \frac{1}{2} [(-1)^3 A^2 + 2A + I] = \begin{bmatrix} 1 & -1 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -1 \end{bmatrix}$$

5. p. 324 #19.

If $k = 2$, the characteristic polynomial is

$$\begin{vmatrix} 0 - t & -a_0 \\ 1 & -a_1 - t \end{vmatrix} = (-1)^2 (t^2 + a_1 t + a_0).$$

Thus the result holds if $k = 2$. Now suppose the formula for the characteristic polynomial is valid if $n = k - 1$. We need to show it for $n = k$. So we expand

$$\begin{vmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} - t \end{vmatrix} = (-t) \begin{vmatrix} -t & 0 & \cdots & 0 & -a_1 \\ 1 & -t & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} - t \end{vmatrix} + (-1)^{k-1} (-a_0) |I_{n-1}|$$

By the induction hypothesis, the determinant of the matrix multiplied by $-t$ is

$$(-1)^{k-1} (a_1 + a_2 t + \cdots + a_{k-1} t^{k-2} + t^{k-1})$$

thus the characteristic polynomial is

$$f_A(t) = (-t) (-1)^{k-1} (a_1 + a_2 t + \cdots + a_{k-1} t^{k-2} + t^{k-1}) + (-1)^{k-1} (-a_0)$$

$$= (-1)^k (a_1 t + a_2 t^2 + \cdots + a_{k-1} t^{k-1} + t^k) + (-1)^k a_0$$

$$= (-1)^k (a_0 + a_1 t + a_2 t^2 + \cdots + a_{k-1} t^{k-1} + t^k).$$

Thus the formula holds when $n = k$ provided it holds for $n = k - 1$. By the principle of mathematical induction, the formula holds for all $n \geq 2$. 