1. (a) A vector space $V$ over a field $F$ is a set on which operations of addition and scalar multiplication are defined so that $x + y \in V$ if $x, y \in V$ and $kx \in V$ if $x \in V$ and $k \in F$ so that the following eight conditions hold.

(i) $x + y = y + x$ for all $x, y \in V$.

(ii) $x + (y + z) = (x + y) + z$ for all $x, y, z \in V$.

(iii) There is a vector $0 \in V$ so that $0 + x = x$ for all $x \in V$.

(iv) For each $x \in V$, there is a vector $y \in V$ so that $x + y = 0$.

(v) $1x = x$ for all $x \in V$.

(vi) $(ab)x = a(bx)$ for all $x \in V$ and all $a, b \in F$.

(vii) $(a + b)x = ax + bx$ for all $x \in V$ and all $a, b \in F$.

(viii) $a(x + y) = ax + ay$ for all $x, y \in V$ and all $a \in F$.

(b) A nonempty subset $S$ of a vector space $V$ is a subspace of $V$ if $S$ itself is a vector space with the operations inherited from $V$.

(c) A set $S \subset V$ is linearly dependent if there exist $u_1, u_2, \ldots, u_n \in S$ and $a_1, \ldots, a_n \in F$ where not all of the scalars are 0 such that

$$a_1 u_1 + a_2 u_2 + \ldots + a_n u_n = 0.$$ 

If $S$ is not linearly dependent, it is called linearly independent.

(d) A basis of a vector space $V$ is a linearly independent subset of $V$ that generates $V$.

(e) The coordinate vector $[v]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ where $a_1, a_2, \ldots, a_n$ are the unique scalars such that $v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$.

(f) Given two vector spaces $V$ and $W$ over the same field, a linear transformation from $V$ to $W$ is a mapping $T : V \to W$ such that

(i) $T(x + y) = T(x) + T(y)$ for all $x, y \in V$.

(ii) $T(kx) = kT(x)$ for all $x \in V$ and all scalars $k$.

(g) Given a linear transformation $T : V \to W$, the null space of $T$ is the set $N(T) = \{ v \in V : T(v) = 0 \}$.

(h) An isomorphism is a linear mapping $T : V \to W$ that is invertible.
(i) The range of a linear transformation \( T : V \rightarrow W \) is the set \( R(T) = \{ w \in W : w = T(v) \text{ for some } v \in V \} \).

2. (a) No, because \((0, 0, 0, 0)\) is not in the set.

(b) Yes, for example \(\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}\), but note that these are linearly dependent.

(c) Yes, since the dimension is 4, a corollary in the text said that any 4 linearly independent vectors must form a basis.

(d) No, by the dimension theorem, the dimension of \( N(T) \) will be at least one, and so \( T \) cannot be one-to-one.

(e) Yes, for example \( T(x, y) = (x, y, 0) \) is a one-to-one mapping from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \). It is not onto.

3. \( S \) is not a subspace with the given operation because for one thing, addition is not commutative. That is,
\[
(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \neq (a_1 + b_1, b_2 - a_2) = (b_1, b_2) + (a_1, a_2).
\]

4. By the criteria for subspaces, we need only check the following three conditions:

(a) \( 0 \in S \) because \( \int_0^1 0 dt = 0 \).

(b) If \( f, g \in S \) then
\[
\int_0^1 (f + g)(t) dt = \int_0^1 f(t) dt + \int_0^1 g(t) dt = 0 + 0 = 0,
\]
hence \( f + g \in S \) and so \( S \) is closed under addition.

(c) If \( f \in S \), then
\[
\int_0^1 kf(t) dt = k \int_0^1 f(t) dt = k \cdot 0 = 0
\]
and so \( kf \in S \). Therefore \( S \) is closed under scalar multiplication.

Together, (a), (b) and (c) show that \( S \) is a subspace.

5. (a) \( T \) is a linear transformation because
\[
T(af + bg) = ((af + bg)(0), (af + bg)(1))
= (af(0) + bg(0), af(1) + bg(1))
= a(f(0), f(1)) + b(g(0), g(1))
= aT(f) + bT(g).
\]

(b) \( T \) is not one-to-one, since, for example, \( T(x) = (0, 1) \) and \( T(x^2) = (0, 1) \). This shows that two different functions in \( C[0, 1] \) map to the same point in \( \mathbb{R}^2 \) and so \( T \) is not one-to-one.
6. Yes, we know that \( P_2(\mathbb{R}) \) has dimension 3 and so when we have a set of three vectors, we need only show that the set is linearly independent to establish that it is a basis. This is what we do now.

\[
a(x^2 + x + 1) + b(x + 1) + c(1) = 0 \iff ax^2 + (a + b)x + (a + b + c)1 = 0 \\
\iff a = 0, a + b = 0 \text{ and } a + b + c = 0 \\
\iff a = 0, b = 0 \text{ and } c = 0.
\]

Therefore, \( \{x^2 + x + 1, x + 1, 1\} \) is a linearly independent set of three vectors in the 3 dimensional space \( P_2(\mathbb{R}) \), therefore it is a basis in \( P_2(\mathbb{R}) \).

7. (a) \( TU \) is defined because the range of \( U \) is in \( \mathbb{R}^3 \) and the domain of \( T \) is \( \mathbb{R}^3 \). The other \( UT \) is not defined because the range of \( T \) is in \( \mathbb{R}^3 \), but the domain of \( U \) is \( \mathbb{R}^2 \).

(b) \( [U] = \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ -2 & 3 \end{bmatrix} \) and \( [T] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 3 & -2 \end{bmatrix} \)

(c) We can find \( TU \) by matrix multiplication, i.e., we know \( [TU] = [T][U] \), and so

\[
[TU] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & 3 \\ 14 & -7 \end{bmatrix}
\]

From this, we can see \( TU(x, y) = \begin{bmatrix} 3 & -4 \\ 1 & 3 \\ 14 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (3x - 4y, x + 3y, 14x - 7y) \).

The alternative way of finding \( TU \) is by direct computation:

\[
TU(x, y) = T(U(x, y)) = T(x - y, 3x, -2x + 3y) \\
= ((x - y) - (-2x + 3y), 3x + (-2x + 3y), x - y + 3(3x - 2(-2x + 3y)) \\
= (3x - 4y, x + 3y, 14x - 7y).
\]

8. Let \( T : V \to W \) be a one-to-one linear transformation, and let \( \{v_1, v_2, \ldots, v_n\} \) be linearly independent in \( V \). To show that \( \{T(v_1), T(v_2), \ldots, T(v_n)\} \) is linearly independent we suppose

\[
a_1T(v_1) + a_2T(v_2) + \ldots + a_nT(v_n) = 0
\]

and we need to show \( a_1 = a_2 = \ldots = a_n = 0 \). Now

\[
a_1T(v_1) + a_2T(v_2) + \ldots + a_nT(v_n) = 0 \implies T(a_1v_1 + a_2v_2 + \ldots + a_nv_n) = 0 \text{ because } T \text{ is linear. This implies}
\]

\[
a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0 \text{ because } T \text{ is one-to-one. This implies}
\]

\[
a_1 = a_2 = \ldots = a_n = 0 \text{ because } \{v_1, \ldots, v_n\} \text{ is lin. ind.}
\]

9. This problem is very similar to Problem #6. First \( V \) has dimension 3 because it has a basis with three vectors. To show that \( \{b_1, b_1 + b_2, b_1 + b_2 + b_3\} \) is a basis of \( V \), we need only show
it is linearly independent. For this observe that,

\[ a_1 b_1 + a_2 (b_1 + b_2) + a_3 (b_1 + b_2 + b_3) = 0 \]
\[ (a_1 + a_2 + a_3) b_1 + (a_2 + a_3) b_2 + a_3 b_3 = 0 \]
\[ a_1 + a_2 + a_3 = 0, \quad a_2 + a_3 = 0, \quad a_3 = 0 \]

which implies

\[ a_1 = a_2 = a_3 = 0 \]

Therefore, \( \{b_1, b_1 + b_2, b_1 + b_2 + b_3\} \) is linearly independent and thus it is a basis of \( V \).

10. To check whether \( x^3 + x^2 + 1 \) is in the span of \( \{x^3 + x^2, x^2 + 1\} \) we need to see if it is a linear combination of the two vectors:

\[ x^3 + x^2 + 1 = a(x^3 + x^2) + b(x^2 + 1) \]

The above implies \( ax^3 + (a + b)x^2 + b = x^3 + x^2 + 1 \). Therefore, \( a = 1, \quad a + b = 1 \) and \( b = 1 \) which is impossible. This shows that \( x^3 + x^2 + 1 \) is not a linear combination of the two vectors hence it is not in their span.