Instructions. Do 10 of the following 11 questions. The last three questions are take-home questions on which you may use MathCAD.

1. (a) Consider the matrix \( A = \begin{bmatrix} 3 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \). Find the eigenvalues of \( A \). Find bases for the eigenspaces corresponding to its eigenvalues. Is the matrix \( A \) diagonalizable? Explain.

(b) Same question with \( A = \begin{bmatrix} 3 & -1 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix} \).

2. Suppose \( A \) is a square matrix.
   (a) Define what is meant by an eigenvalue with corresponding eigenvector for the matrix \( A \).
   (b) Explain or prove why the eigenvalues of \( A \) are found by solving the equation \( |A - \lambda I| = 0 \).
   (c) Suppose \( A \) is an invertible matrix. Is it possible for 0 to be an eigenvalue of \( A \)? Justify your answer.

3. Suppose the determinant of \( \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \) is 5. Find the determinants of
   (a) \( A = \begin{bmatrix} a & b & c \\ a & b & c \\ 10g & 10h & 10i \end{bmatrix} \)
   (b) \( B = \begin{bmatrix} d & e & f \\ 10a & 10b & 10c \\ g + 10d & h + 10e & i + 10f \end{bmatrix} \)
   (c) \( B = \begin{bmatrix} 10d & 10e & 10f \\ 10a & 10b & 10c \\ 10g & 10h & 10i \end{bmatrix} \)

4. Let \( A = \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & 1 \\ -1 & 1 & 7 \end{bmatrix} \), find the inverse of \( A \), and write \( A \) and \( A^{-1} \) as products of elementary matrices.

5. Let \( A, B \) and \( C \) be \( n \) by \( n \) matrices where \( A = BCB^{-1} \).
   (a) Show that \( A^n = BC^nB^{-1} \).
   (b) Suppose that \( B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \) and \( C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \). Find a formula for \( A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).
6. (a) Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors 
\[ \{(1, 1, 1, 1), (1, 2, 3, 4), (5, 6, 7, 8), (9, 10, 11, 12)\}. \]
Write each vector in the original set as a linear combination of the basis vectors you found.
(b) Is the vector $(1, 3, 4, 5)$ in the span of the vectors listed in (a)?

7. Prove that $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$.

8. Use Cramer’s rule to write formulas for the solutions $x$ and $y$ to the system
\[
\begin{align*}
ax + by &= e \\
fx + dy &= f
\end{align*}
\]
where $ad - bc \neq 0$, and illustrate your formula with the system
\[
\begin{align*}
2x - y &= 3 \\
5x + 4y &= 4
\end{align*}
\]

**Part II.** The following three questions may be completed with the help of MathCAD. They are due by 8am tomorrow morning. You may not ask anyone for assistance on these questions. You may use your notes, book or other resources.

9. Consider the system of equations
\[
\begin{align*}
x + 2y + 2z &= 5 \\
-2x - 5y - 2z &= 8 \\
2x + 4y + 7z &= 19
\end{align*}
\]
Solve this system using each of the following three methods:
(a) by row reducing the augmented matrix;
(b) using the inverse of the coefficient matrix;
(c) using Cramer’s rule.

10. Find a formula for $x_n$ defined by the recursion relation $x_n = 5x_{n-1} + 6x_{n-2}$ where $x_1 = 0$ and $x_2 = 1$.

11. (a) Verify that $\beta = \{(1,1,1,1),(-1,2,2,2),(-2,-2,4,4),(-3,-3,-3,6)\}$ is a basis of $\mathbb{R}^4$.
(b) Given the ordered basis in (a), find $x$ if its coordinate vector is $[x]_\beta = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$.
(c) Given the ordered basis in (a), find $[x]_\beta$ when $x = (-6,3,15,24)$. Verify by hand that the coordinate vector you found is correct.